Lattice QCD calculation of Hyperon transition form factors using the Feynman-Hellmann method.

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The University of Adelaide

Jefferson Lab Theory Seminar
31 October 2022

CKM Matrix

$$\begin{bmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{bmatrix} = \begin{bmatrix} 0.97401(11) & 0.22650(48) & 0.00361(11) \\ 0.22636(48) & 0.97320(11) & 0.04053(83) \\ 0.00854(23) & 0.03978(82) & 0.99917(04) \end{bmatrix}$$

- SM requires unitarity: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$

PDG Review of particle physics 2020

- $|V_{us}|$ can be constrained by
  - (Semi-) leptonic Kaon decays
  - Semi-leptonic hyperon decays
- Transition matrix element for semi-leptonic hyperon decay:

$$T = \frac{G_F}{\sqrt{2}} V_{us} \left[\langle B' | \bar{u} \gamma_\mu \gamma^5 s | B \rangle - \langle B' | \bar{u} \gamma_\mu s | B \rangle \right] \bar{l} \gamma^\mu (1 - \gamma^5) \nu_l$$

From experimental decay widths

Axial-vector

Vector

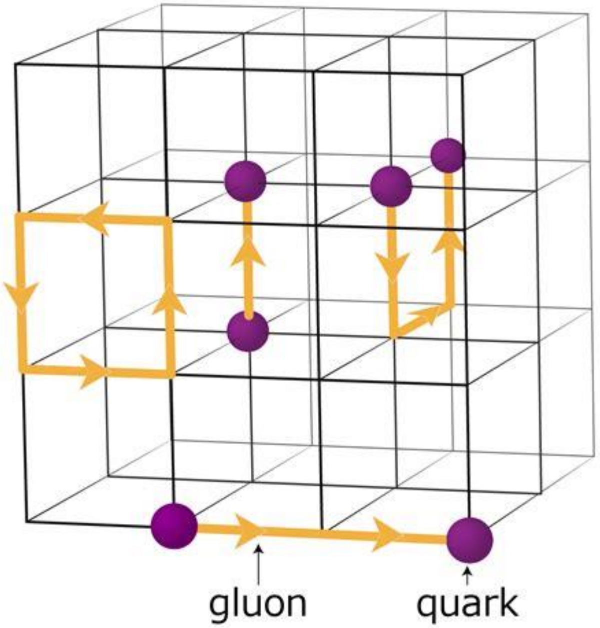
- Lattice QCD determinations can improve on phenomenological form factor values

How Does Lattice QCD Work?

- Discretize space-time
  - Quarks on the lattice sites
  - Gauge fields on the links
- Use Monte Carlo sampling to generate gauge fields
- Calculate expectation value by averaging over Gauge configurations

$$\langle\Omega|\mathcal{O}|\Omega\rangle = \frac{1}{Z}\int D\bar{\psi}\, D\psi\, D A\, \mathcal{O}\, e^{iS}$$

$$\xrightarrow[\text{Euclideanise}]{\text{Discretise}} \frac{1}{N}\sum_{i=1}^{N} \mathcal{W}\{\mathcal{O}\}\,(U_i)$$



- Then account for the systematics (finite volume, lattice spacing, larger-than-physical pion mass)

What can we calculate with Lattice QCD?

Nucleon Mass

- Creation operator which couples to nucleons:
  - Will couple to any state with the same quantum numbers
- The spectral decomposition of the correlation function includes a tower of states with increasing energies
- Effective energy of the correlator asymptotes towards the ground state energy

$$C^{2pt}(x,0) = \langle T\{ \underbrace{\chi_N(x,t)}_{\text{Annihilate nucleon at the sink}} \; \underbrace{\bar{\chi}_N(0)}_{\text{Create nucleon at the source}} \}\rangle$$

$$C^{2pt}(t) = \sum_{i=0} A_i e^{-E_i(\vec{p})t}$$

$$E_{\text{eff}} = \ln\left(\frac{C(t)}{C(t+1)}\right) \xrightarrow{t \gg 0} E_0$$

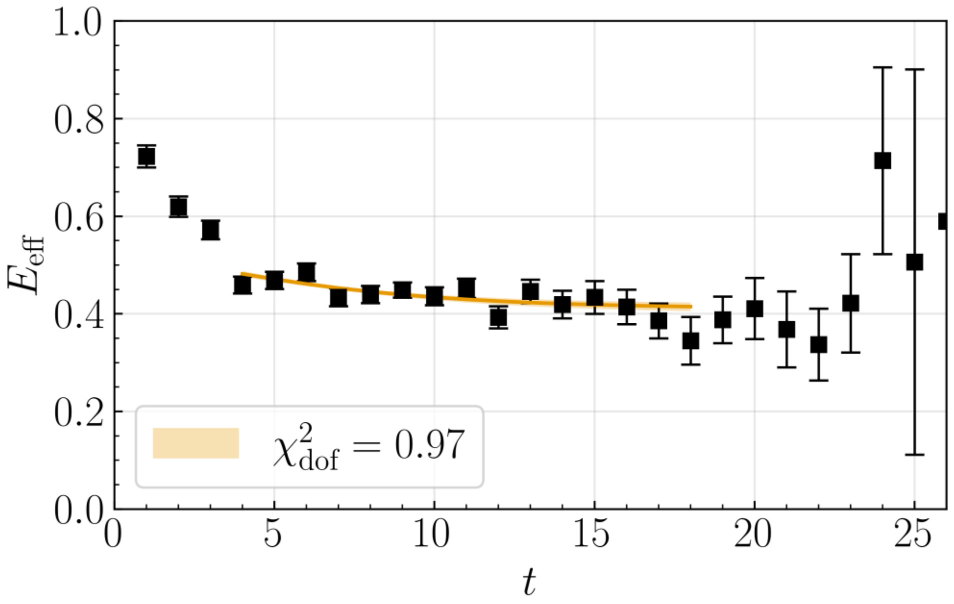
What can we calculate with Lattice QCD?

Nucleon Mass

- Calculate correlation function on each gauge configuration
- Use bootstrap resampling to get uncertainties

- Fit the correlator using exponential function ansatz
  - Signal to Noise ratio decreases at large time
  - Ground state dominates signal only at large times
  - Signal strength decreases at large momenta

$$C^{2pt}(t) = \sum_{i=0} A_i e^{-E_i(\vec{p})t}$$



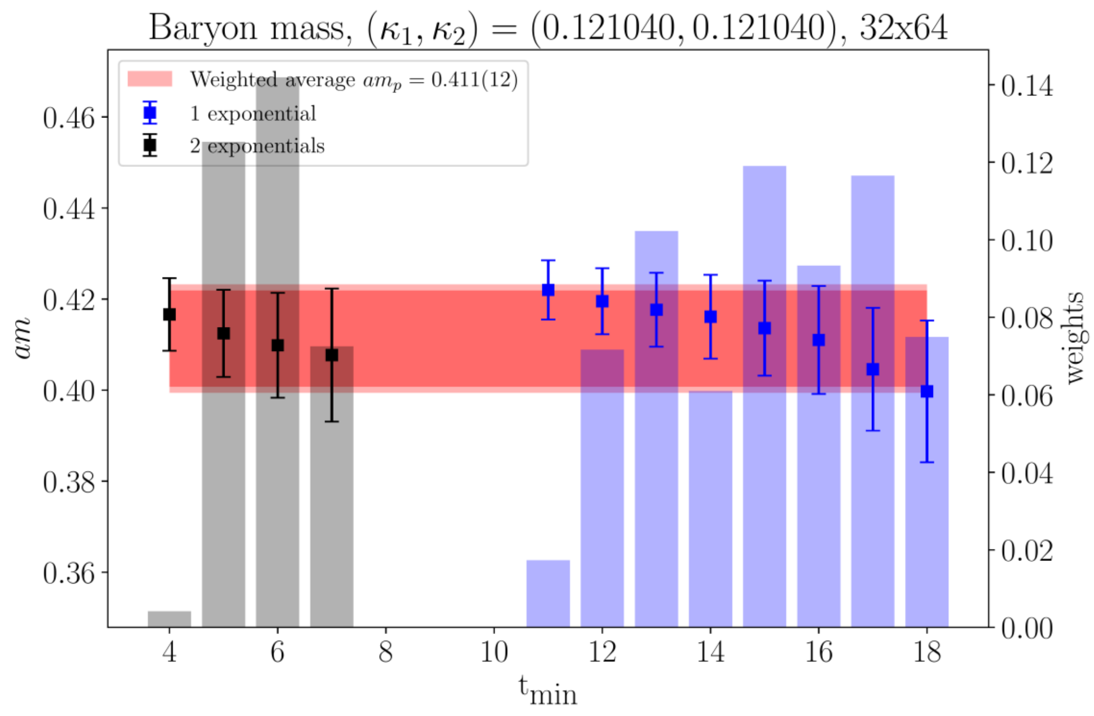
What can we calculate with Lattice QCD?

How can we deal with these issues:

- Include excited states in the ansatz
  - Can fit from earlier time
- Use multiple operators to improve overlap with the ground state
  - Signal improves
  - Easier to filter out excited states
- Weighted averaging over multiple fit ranges
  - Reduces effects of researcher's fit window choice

$$C_0^{2pt}(t) = A_0 e^{-E_0 t}$$

$$C_1^{2pt}(t) = A_0 e^{-E_0 t} + A_1 e^{-E_1 t}$$



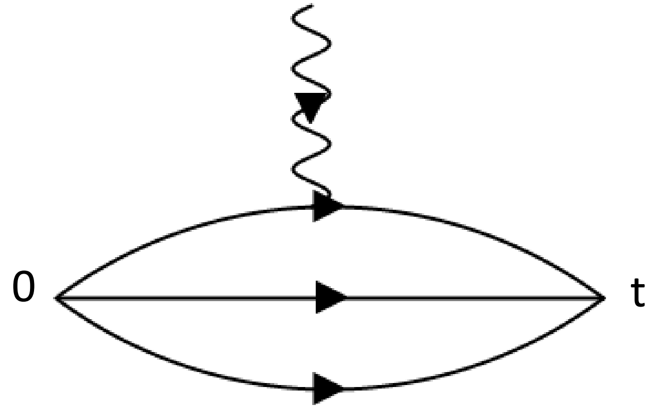
What can we calculate with Lattice QCD?

Baryon Matrix Elements

- Include a current insertion operator between creation and annihilation

$$C^{3pt}(t) = \Big\langle \underbrace{\chi_N(x_2,t)}_{\text{Annihilate nucleon at the sink}} \; \underbrace{\hat{\mathcal{O}}(x_1,\tau)}_{\text{Insert current}} \; \underbrace{\bar{\chi}_N(0,0)}_{\text{Create nucleon at the source}} \Big\rangle$$

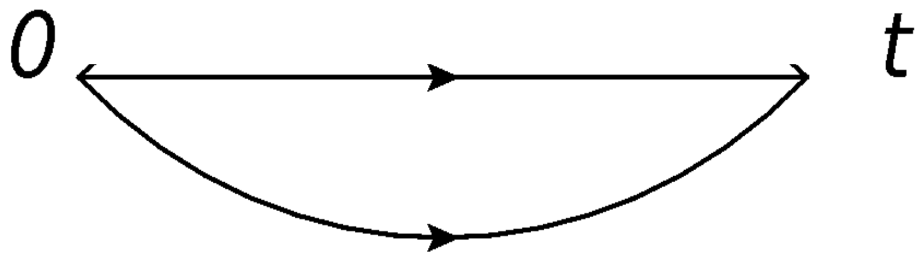
$$C^{3pt}(t) = \sum_{B,B'} \underbrace{e^{-E_{B'}(t-\tau)} e^{-E_B \tau}}_{\text{Two towers of exponentially decaying excited states}} \langle\Omega|\chi(0)|B'\rangle \underbrace{\langle B'|\mathcal{O}|B\rangle}_{\text{Form factors are contained in this matrix element}} \langle B|\bar{\chi}(0)|\Omega\rangle$$



Three-point functions

On the lattice:

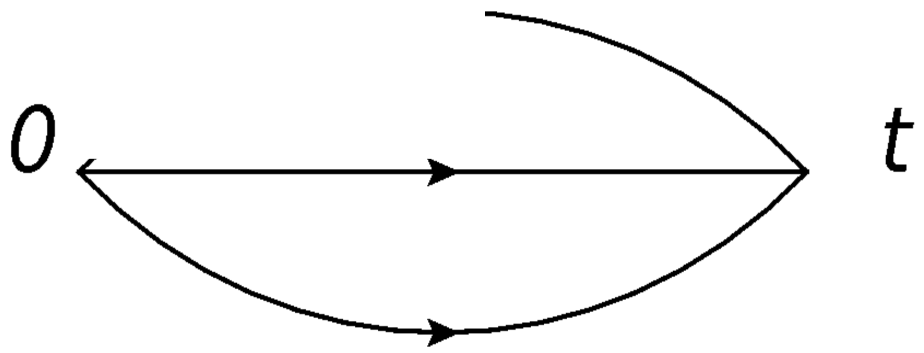
- Use a sequential source
- Fix the sink time and momentum



Three-point functions

On the lattice:

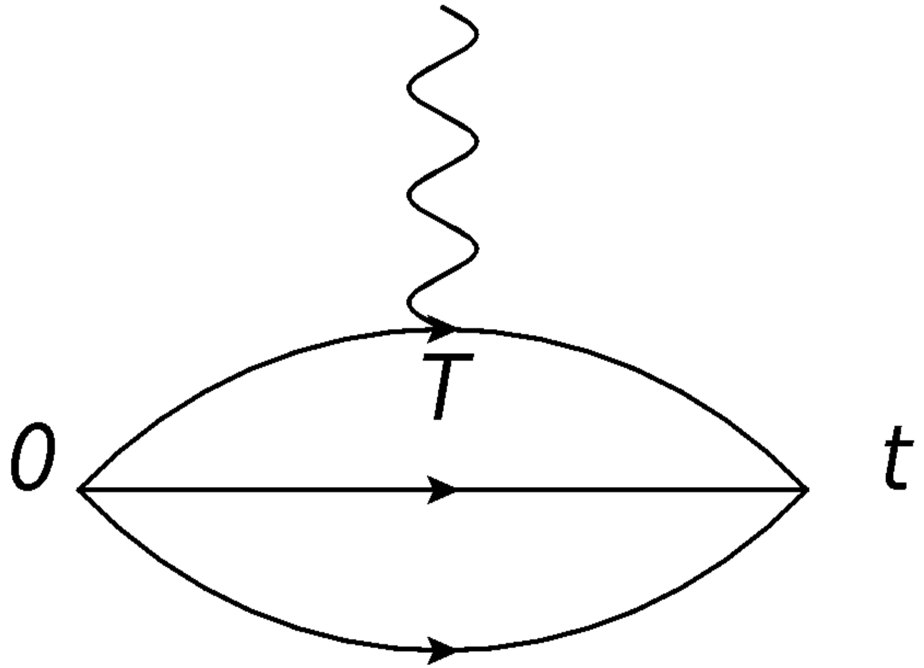
- Use a sequential source
- Fix the sink time and momentum
- Invert from the sequential source at the sink to the operator insertion



Three-point functions

On the lattice:

- Use a sequential source
- Fix the sink time and momentum
- Invert from the sequential source at the sink to the operator insertion
- Insert the current operator and connect it with the source



Three-point functions

- Advantages:
  - Allows for operator choice after all the inversions
  - Free choice of operator, momentum transfer
- Disadvantages:
  - It has two time windows for which excited states need to be controlled
  - Requires separate inversions for
    - Every sink momentum
    - Every source-sink time separation
    - Every polarisation

Feynman-Hellmann method

1. Modify the QCD action with an operator

$$\mathcal{L} \to \mathcal{L} + \lambda \mathcal{O}$$

2. Calculate the energy spectrum with the modified action
3. Relate the change in energy to the matrix element:

$$\left.\frac{\partial E_X}{\partial \lambda}\right|_{\lambda=0} \propto \langle X|\mathcal{O}|X\rangle$$

Connected contributions => Invert the new fermion matrix

Disconnected contributions => generate new gauge configurations

Non-forward Feynman-Hellmann method

1. Modify the QCD action with an operator including momentum

$$\mathcal{L} \to \mathcal{L} + \lambda \left(e^{i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}}\right) \mathcal{O}$$

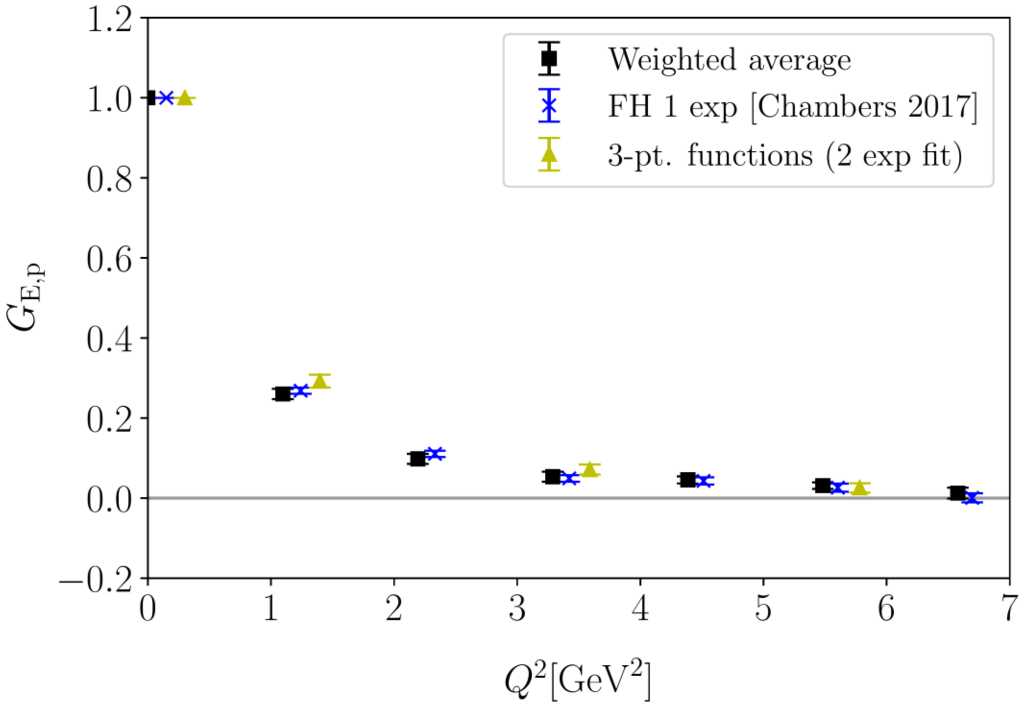
2. Calculate energy spectrum
3. Relate change in energy to the matrix element

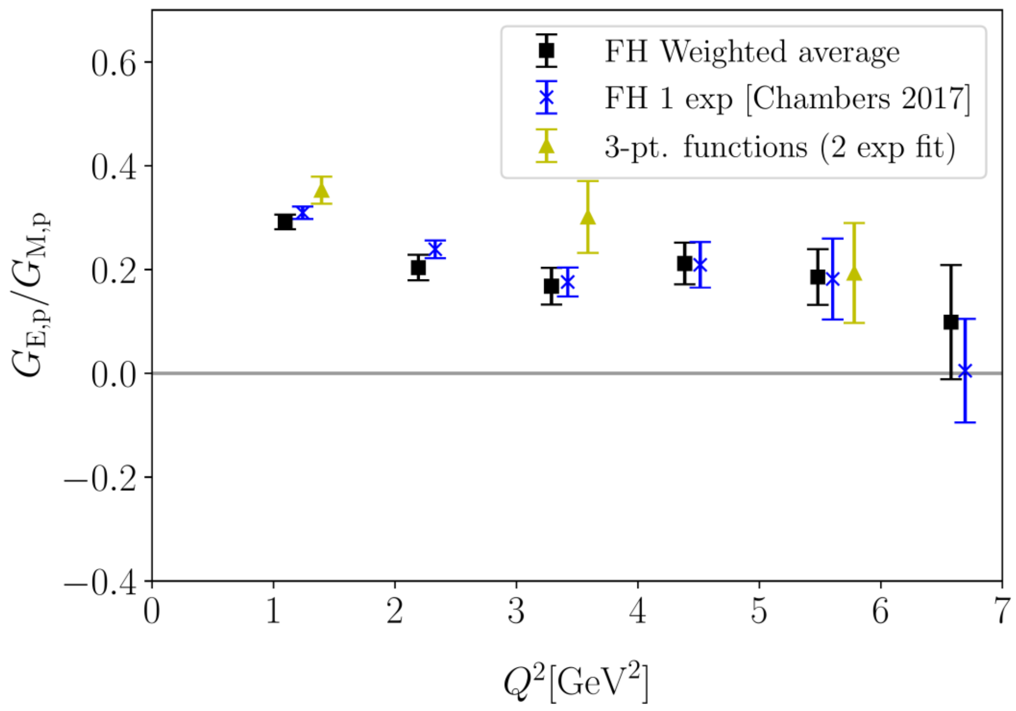
$$\left.\frac{\partial E_X(\vec{p}\,')}{\partial \lambda}\right|_{\lambda=0} \propto \langle X(p')|\mathcal{O}|X(p)\rangle$$

This requires Breit frame kinematics: $E_X(\vec{p}\,') = E_X(\vec{p})$

Non-forward Feynman-Hellmann method results

Matrix elements up to high momentum are accessible:





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Transition Form Factors

$$\langle B'|\mathcal{V}_\mu|B\rangle = \gamma_\mu f_1^{BB'}(Q^2) + \sigma_{\mu\nu}q_\nu \frac{f_2^{BB'}(Q^2)}{M_B + M_{B'}} - iq_\mu \frac{f_3^{BB'}(Q^2)}{M_B + M_{B'}}$$

$f_1(Q^2=0)$ can be used to constrain V_{us} element of the CKM matrix

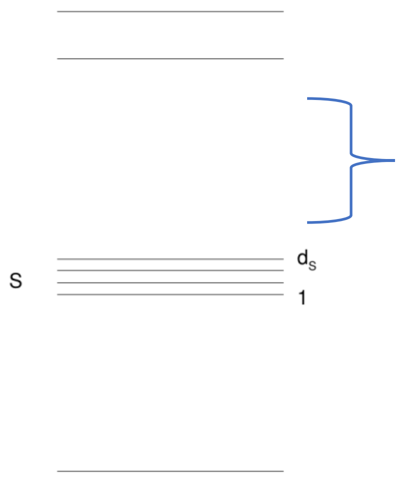
=0 if B=B'

Breit frame condition cannot be satisfied as easily anymore

=> Consider quasi-degenerate energy states

$$E_{B_r}(\vec{p}_r) = \bar{E} + \epsilon_r\,, \qquad r = 1, \ldots, d_S$$

d_S states which are quasi-degenerate in energy:



quasi-degenerate states must be well separated from any other states

Feynman-Hellmann For Quasi-Degenerate Energies

Consider a two-point function with a Hamiltonian which includes a perturbing operator:

$$C_{\lambda B'B}(t;\vec{p},\vec{q}) = {}_\lambda\langle 0|\hat{\tilde{B}}'(0;\vec{p}\,')\underbrace{\hat{S}_\lambda(\vec{q}\,)}^{t}\hat{\tilde{B}}(0,\vec{0}\,))|0\rangle_\lambda$$

Transfer matrix with perturbed Hamiltonian:

$$\hat{S}_\lambda(\vec{q}) = e^{-\hat{H}_\lambda(\vec{q})}$$

$$\hat{H}_\lambda(\vec{q}) = \hat{H}_0 - \lambda\hat{\tilde{O}}(\vec{q})$$

Insert two complete sets-of-states into the two-point function:

$$C_{\lambda B'B}(t;\vec{p},\vec{q}) = \sum_{X(\vec{p}_X)}\sum_{Y(\vec{p}_Y)} {}_\lambda\langle 0|\hat{\tilde{B}}'(\vec{p}\,')|X(\vec{p}_X)\rangle \underbrace{\langle X(\vec{p}_X)|\hat{S}_\lambda(\vec{q}\,)^t|Y(\vec{p}_Y)\rangle}_{\text{We want this}} \langle Y(\vec{p}_Y)|\hat{\tilde{B}}(\vec{0}\,)|0\rangle_\lambda$$

Dyson expansion

$$\langle X(\vec{p}_X)|\hat{S}_\lambda(\vec{q})^t|Y(\vec{p}_Y)\rangle$$

Expand the transfer matrix for small values of λ:

$$e^{-(\hat{H}_0-\lambda\hat{\tilde{\mathcal{O}}})t} = e^{-\hat{H}_0 t} + \lambda\int_0^t dt' e^{-\hat{H}_0(t-t')}\,\hat{\tilde{\mathcal{O}}}\,e^{-\hat{H}_0 t'} + \lambda^2(\text{compton terms})$$

Consider the separate pieces:

$$\begin{aligned}
\langle B_r|e^{-(\hat{H}_0-\lambda\hat{\tilde{\mathcal{O}}})t}|B_s\rangle &= e^{-\bar{E}t}\left(\delta_{rs} + tD_{rs} + O(2)\right) \\
\langle B_r|e^{-(\hat{H}_0-\lambda\hat{\tilde{\mathcal{O}}})t}|Y\rangle &= e^{-\bar{E}t}\left(\lambda\frac{\langle B_r|\hat{\tilde{\mathcal{O}}}|Y\rangle}{E_Y - E_{B_r}} + O(2)\right) + \begin{matrix}\text{more}\\ \text{damped}\end{matrix}
\end{aligned}$$

D_{rs} is defined as:

$$D_{rs}(\lambda,\epsilon) = -\epsilon_e\delta_{rs} + \lambda\,\langle B_r(\vec{p}_r)|\hat{\tilde{\mathcal{O}}}(\vec{q})|B_s(\vec{p}_s)\rangle$$

Diagonalisation of $D_{rs}(\lambda, \epsilon) = -\epsilon_e \delta_{rs} + \lambda \langle B_r(\vec{p}_r) | \hat{\tilde{\mathcal{O}}}(\vec{q}) | B_s(\vec{p}_s) \rangle$

We can diagonalise the matrix D, such that:

$$D_{rs} = \sum_{i=1}^{d_S} \mu^{(i)} e_r^{(i)} e_s^{(i)\,*}$$

This allows us to write the two-point function as:

$$C_{\lambda\, B'B}(t; \vec{p}, \vec{q}) = \sum_{i=1}^{d_S} A^{(i)}_{\lambda\, B'B}(\vec{p}, \vec{q})\, e^{-E_\lambda^{(i)}(\vec{p},\vec{q})t}$$

With the energies being determined by the eigenvalues

$$E_\lambda^{(i)}(\vec{p}, \vec{q}) = \bar{E}(\vec{p}, \vec{q}) - \mu^{(i)}(\epsilon, \lambda; \vec{p}, \vec{q}), \quad i = 1, \ldots, d_S$$

The problem is now a Generalized Eigenvalue Problem win B_r, B_s space to find $E_\lambda^{(i)}(\vec{p}, \vec{q})$

$\Sigma^- \to n$ Transition

- Consider the action: $S = S_g + \int_x (\bar{u}, \bar{s}) \begin{pmatrix} D_u & -\lambda\mathcal{T} \\ -\lambda\mathcal{T}' & D_s \end{pmatrix} \begin{pmatrix} u \\ s \end{pmatrix} + \int_x \bar{d}\, D_d\, d$

  where $\mathcal{T}(x, y; \vec{q}) = \gamma e^{i\vec{q}\cdot\vec{x}} \delta_{x,y}$

- Construct a matrix of correlation functions:

$$C_{\lambda B'B} = \begin{pmatrix} C_{\lambda\Sigma\Sigma} & C_{\lambda\Sigma N} \\ C_{\lambda N\Sigma} & C_{\lambda NN} \end{pmatrix}_{B'B}$$

- After solving the GEVP, the energies are related to the matrix element

$$\Delta E_{\lambda\Sigma N} = \sqrt{(E_N(\vec{q}) - M_\Sigma)^2 + 4\lambda^2 \left| \langle N(\vec{q})|\bar{u}\gamma_4 s|\Sigma(\vec{0})\rangle \right|^2}$$

$\Sigma^- \to n$ Transition

$$C_{\lambda B'B} = \begin{pmatrix} C_{\lambda\Sigma\Sigma} & C_{\lambda\Sigma N} \\ C_{\lambda N\Sigma} & C_{\lambda NN} \end{pmatrix}_{B'B}$$

- Each correlator is built from a Green's function:

$$\begin{pmatrix} G_{uu} & G_{us} \\ G_{su} & G_{ss} \end{pmatrix} = \begin{pmatrix} (\mathcal{M}^{-1})_{uu} & (\mathcal{M}^{-1})_{us} \\ (\mathcal{M}^{-1})_{su} & (\mathcal{M}^{-1})_{ss} \end{pmatrix}$$

- Where:

$$\begin{aligned} G^{(uu)} &= (1 - \lambda^2 D_u^{-1} \mathcal{T} D_s^{-1} \gamma_5 \mathcal{T}^\dagger \gamma_5)^{-1} D_u^{-1} \\ G^{(ss)} &= (1 - \lambda^2 D_s^{-1} \gamma_5 \mathcal{T}^\dagger \gamma_5 D_u^{-1} \mathcal{T})^{-1} D_s^{-1} \end{aligned}$$

→ Problem: inversion within another inversion

$$\begin{aligned} G^{(us)} &= \lambda D_u^{-1} \mathcal{T} G^{(ss)}\,, \\ G^{(su)} &= \lambda D_s^{-1} \gamma_5 \mathcal{T}^\dagger \gamma_5 G^{(uu)} \end{aligned}$$

Iterative Green's functions

$$\begin{pmatrix} G_{uu} & G_{us} \\ G_{su} & G_{ss} \end{pmatrix}$$

- We can expand the Green's functions for small lambda

$$\begin{aligned} G^{(uu)}_{2n+2} &= D_u^{-1} + \lambda^2 D_u^{-1}\mathcal{T} D_s^{-1}\gamma_5\mathcal{T}^\dagger\gamma_5 G^{(uu)}_{2n}\,, \\ G^{(ss)}_{2n+2} &= D_s^{-1} + \lambda^2 D_s^{-1}\gamma_5\mathcal{T}^\dagger\gamma D_u^{-1}\mathcal{T} G^{(ss)}_{2n}\,, \end{aligned}$$

- Will give the exact result as n goes to infinity.
- We will consider up to order $\mathcal{O}(\lambda^4)$

$$G^{(us)}_{2n+1} = \lambda D_u^{-1}\mathcal{T} G^{(ss)}_{2n}$$

$$G^{(su)}_{2n+1} = \lambda D_s^{-1}\gamma_5\mathcal{T}^\dagger\gamma_5 G^{(uu)}_{2n}$$

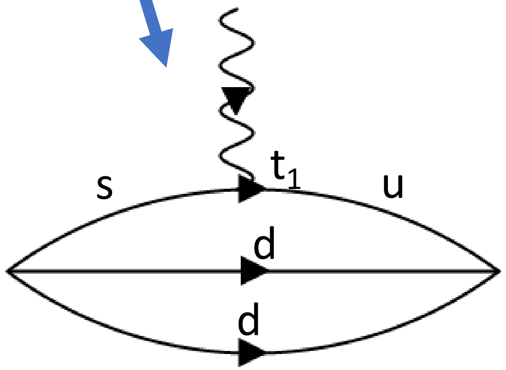
- This allows changing the value of lambda after the inversions!

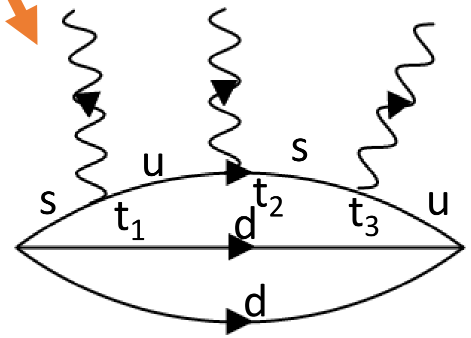
Iterative Green's functions

$$\begin{pmatrix} G_{uu} & G_{us} \\ G_{su} & G_{ss} \end{pmatrix}$$

For example, take $\mathcal{O}(\lambda^3)$:

$$G_3^{(us)} = \lambda D_u^{-1} \mathcal{T} D_s^{-1} + \lambda^3 D_u^{-1} \mathcal{T} D_s^{-1} \gamma_5 \mathcal{T}^\dagger \gamma_5 D_u^{-1} \mathcal{T} D_s^{-1}$$

$G_3^{(us)}$ = $\sum_{t_1} \lambda$ 

+ $\sum_{t_1, t_2, t_3} \lambda^3$ 

Kinematics

- Choose the Sigma to be at rest and change the momentum of the nucleon
- Choose the operator to be the vector current γ_4
- For hyperons with quasi-degenerate energies the shift due to a perturbation in the action lambda is

$$\Delta E_{\lambda \Sigma N} = \sqrt{\left(E_N(\vec{q}) - M_\Sigma\right)^2 + 4\lambda^2 \left| \langle N(\vec{q}) | \bar{u}\gamma_4 s | \Sigma(\vec{0}) \rangle \right|^2}$$

- When the energy of the nucleon equals the mass of the Sigma, this will be linear in lambda.
  - How far from the degenerate energy point can we make this work?
    - At Q^2=0?

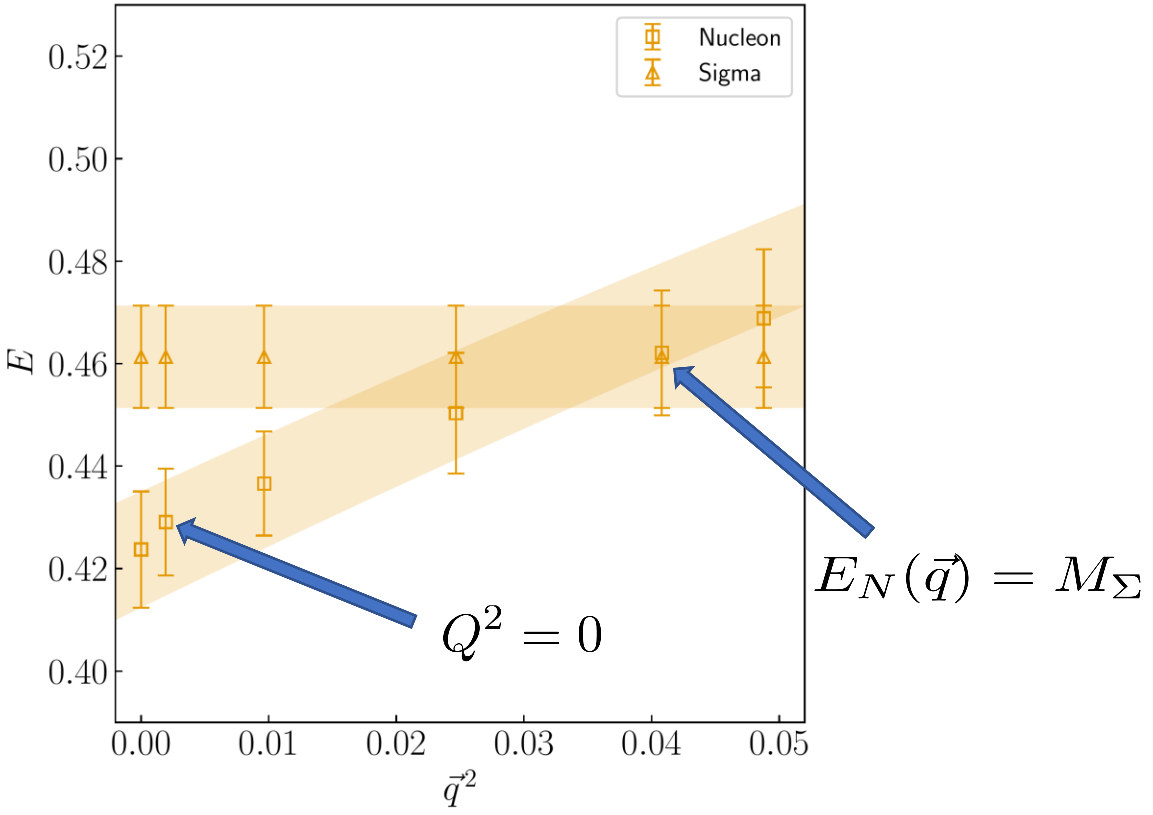
(Partially) Twisted Boundary Conditions

$$Q^2 = -(M_\Sigma - E_N(\vec{q}))^2 + \vec{q}^2$$

- Momentum on the lattice is quantised
  - how do we get to the energy-degenerate point?
- Twisted boundary conditions add a complex phase to the boundary conditions
  - Gives lattice correlators any momentum

$$q(\vec{x} + N_s\vec{e}_i, t) = e^{i\theta_i} q(\vec{x}, t)$$

- Q^2 points of interest:
  - Degenerate energy $E_N(\vec{q}) = M_\Sigma$
  - $Q^2 = 0$



Lattice details

- 32^3x64 lattice size
- Lattice spacing a=0.074fm
- N_f = 2 + 1, O(a)-improved clover Wilson fermions
- Up and down quark are degenerate
- O(500) configurations used for each choice of momentum

run #	θ_2/π	$\vec{q}^{\,2}$	E_N	$M_\Sigma - E_N$	$Q^2[\mathrm{GeV}^2]$
1	0.0	0.0	0.424(11)	0.0366(33)	-0.0095
2	0.448	0.0019	0.429(10)	0.0351(35)	0.0048
3	1	0.0096	0.437(10)	0.0301(42)	0.0620
4	1.6	0.0247	0.450(12)	0.0182(57)	0.1732
5	2.06	0.0408	0.462(12)	0.0030(69)	0.2901
6	2.25	0.0488	0.469(13)	-0.0037(78)	0.3472

Generalized EigenValue Problem (GEVP)

- Diagonalise the matrix

$$C_{\lambda B'B} = \begin{pmatrix} C_{\lambda\Sigma\Sigma} & C_{\lambda\Sigma N} \\ C_{\lambda N\Sigma} & C_{\lambda NN} \end{pmatrix}_{B'B}$$

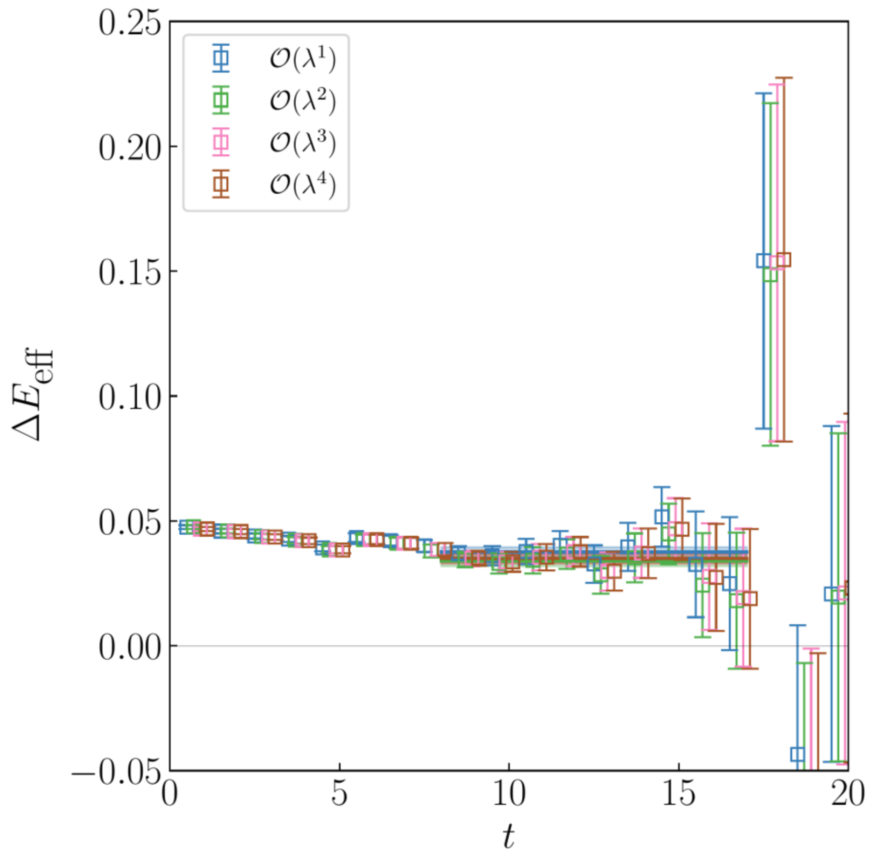
  - Gives two eigenvectors and eigenvalues
  - Eigenvalues related to the energy

- Use the eigenvectors to project out two correlation functions:

$$C_\lambda^{(i)}(t) = v^{(i)\,\dagger} C_\lambda(t) u^{(i)}\,, \quad i = \pm$$

- Take the ratio of the two correlators and fit to the energy shift ΔE.

$$R_\lambda(t;\vec{0},\vec{q}) = \frac{C_\lambda^{(-)}(t;\vec{0},\vec{q})}{C_\lambda^{(+)}(t;\vec{0},\vec{q})} \overset{t\gg 0}{\propto} e^{-\Delta E_\lambda(\vec{0},\vec{q})t}$$



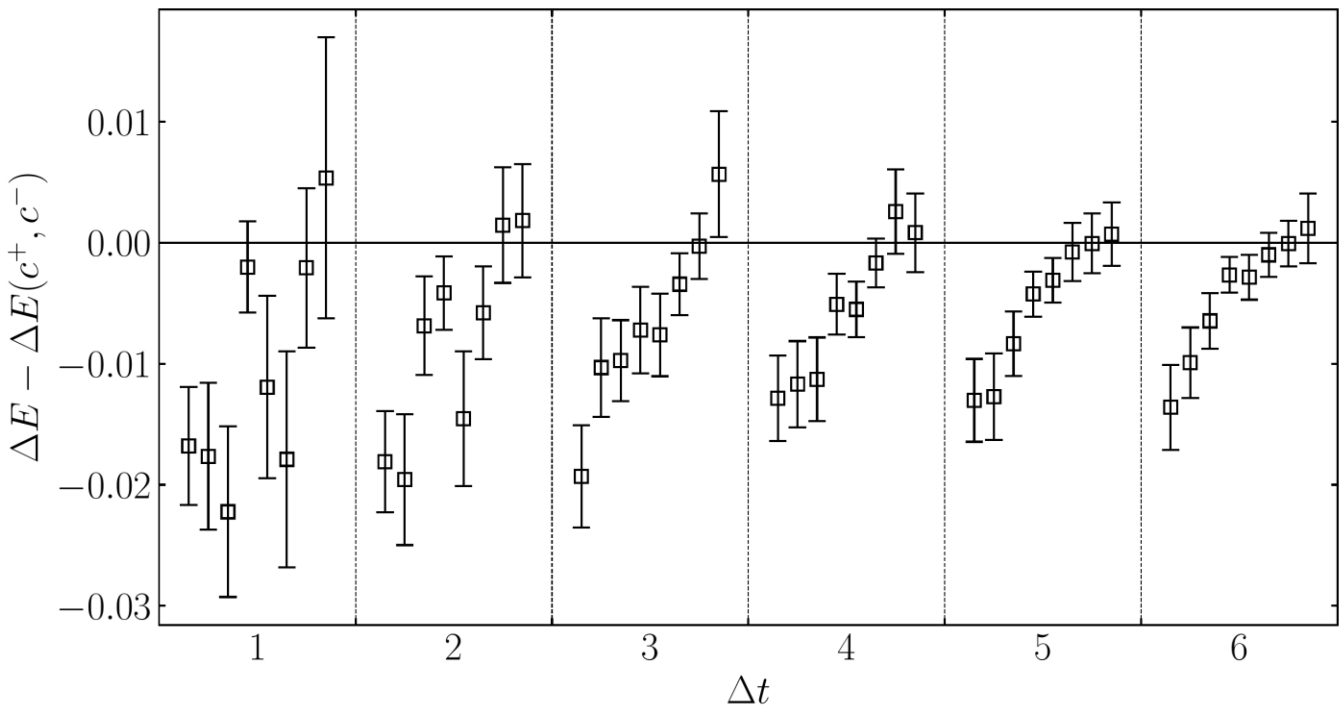
GEVP Stability

GEVP depends on two parameters (t_0 & Δt):

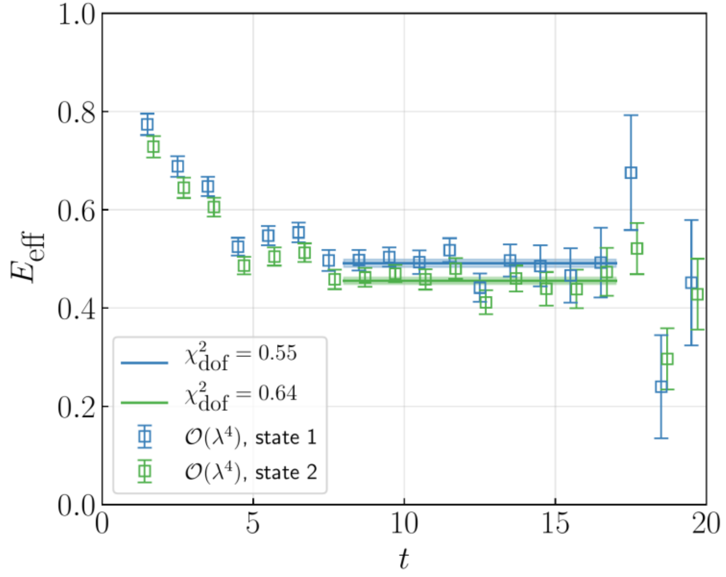
$$C_\lambda^{-1}(t_0)C_\lambda(t_0+\Delta t)e^{(i)} = c^{(i)}e^{(i)}$$

Stable under GEVP parameters?

- Do the GEVP for many value of t_0 and Δt
- Calculate the value of $\Delta E(c^+, c^-)$ from the eigenvalues
- Compare with ΔE from the fit to the ratio of correlators
- For each Δt we show results from t_0=1-8



Ground state saturated:

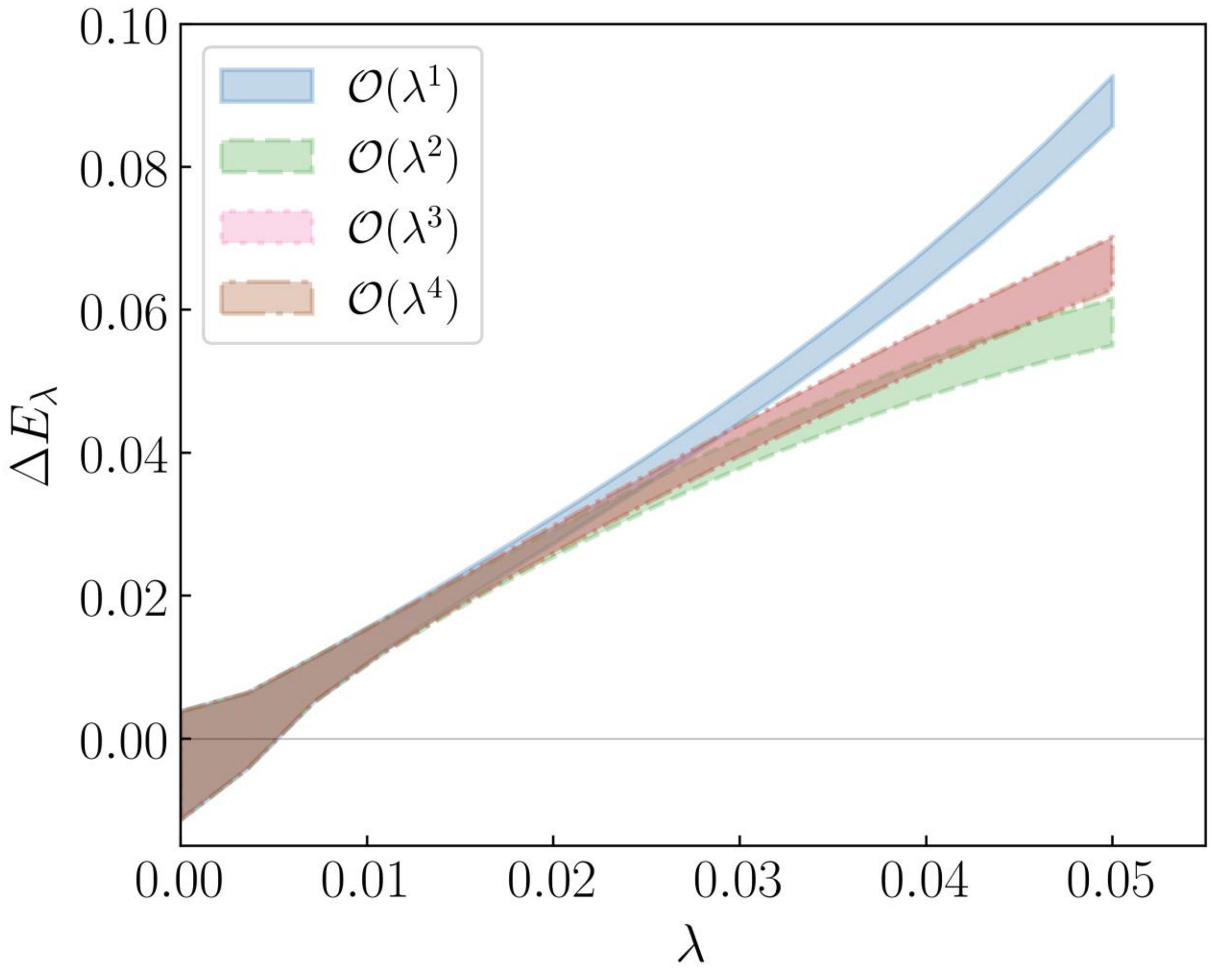


Result from GEVP are stable in range $\Delta t \geq 4$ and $t_0 \geq 6$

ΔE as a function of λ when $E_N(\vec{q}) = M_\Sigma$

$$R_\lambda(t;\vec{0},\vec{q}) = \frac{C_\lambda^{(-)}(t;\vec{0},\vec{q})}{C_\lambda^{(+)}(t;\vec{0},\vec{q})} \overset{t\gg 0}{\propto} e^{-\Delta E_\lambda(\vec{0},\vec{q})t}$$

- Iterative Method: higher orders in lambda increase the range over which our approximation holds
- We want to fit in the region where the dependence is linear
- Choose the region where the two highest order results agree
  - O(λ^3) and O(λ^4) agree up to λ =0.05



$$\Delta E_{\lambda\Sigma N} = \sqrt{(E_N(\vec{q}) - M_\Sigma)^2 + 4\lambda^2 \left| \langle N(\vec{q})|\bar{u}\gamma_4 s|\Sigma(\vec{0})\rangle \right|^2}$$

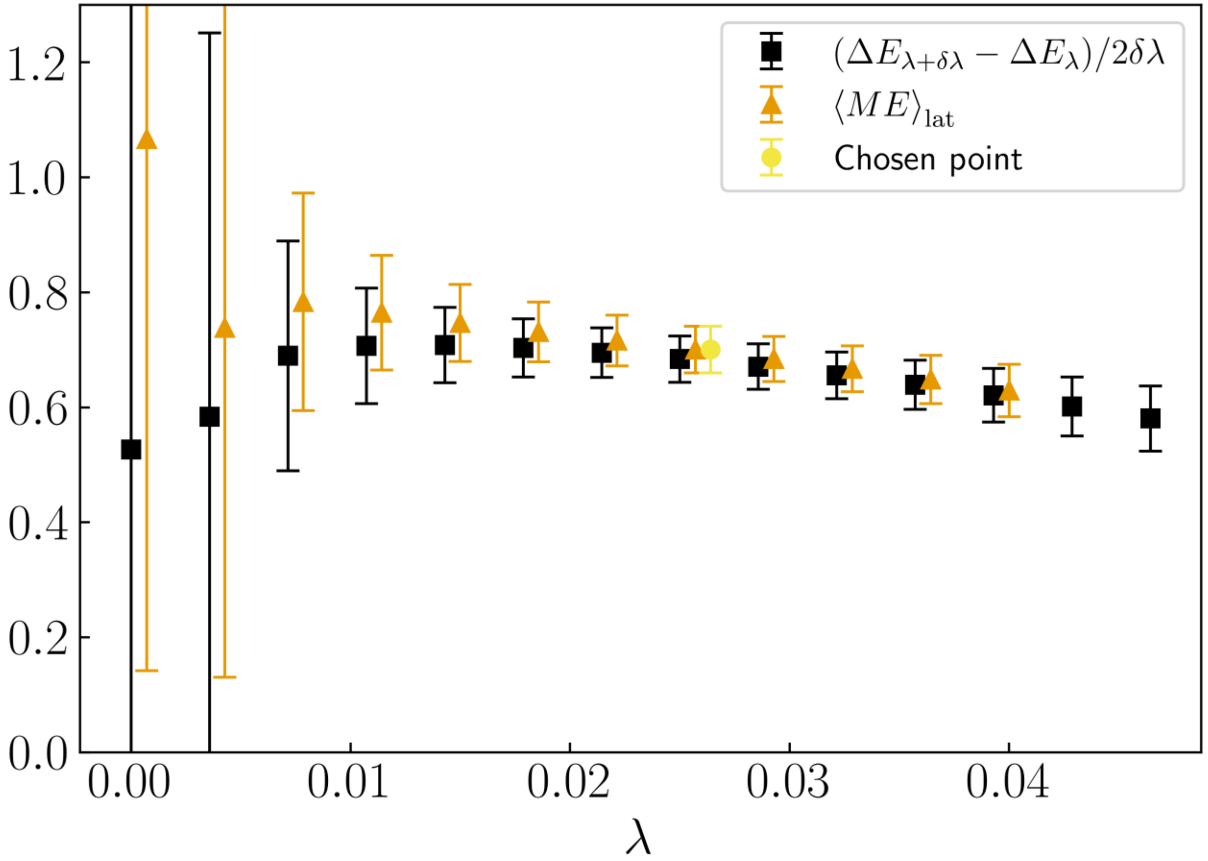
Fitting the slope of ΔE

$$\Delta E_\lambda = \sqrt{(E_N(\vec{q}) - M_\Sigma)^2 + 4\lambda^2 \langle ME \rangle_{\text{lat}}^2}$$

- Consider the ratio at two values of λ close to each other.

$$\frac{R_{\lambda+\delta\lambda}(t)}{R_\lambda(t)} \overset{t\gg 0}{\propto} e^{-(\Delta E_{\lambda+\delta\lambda} - \Delta E_\lambda)t}$$

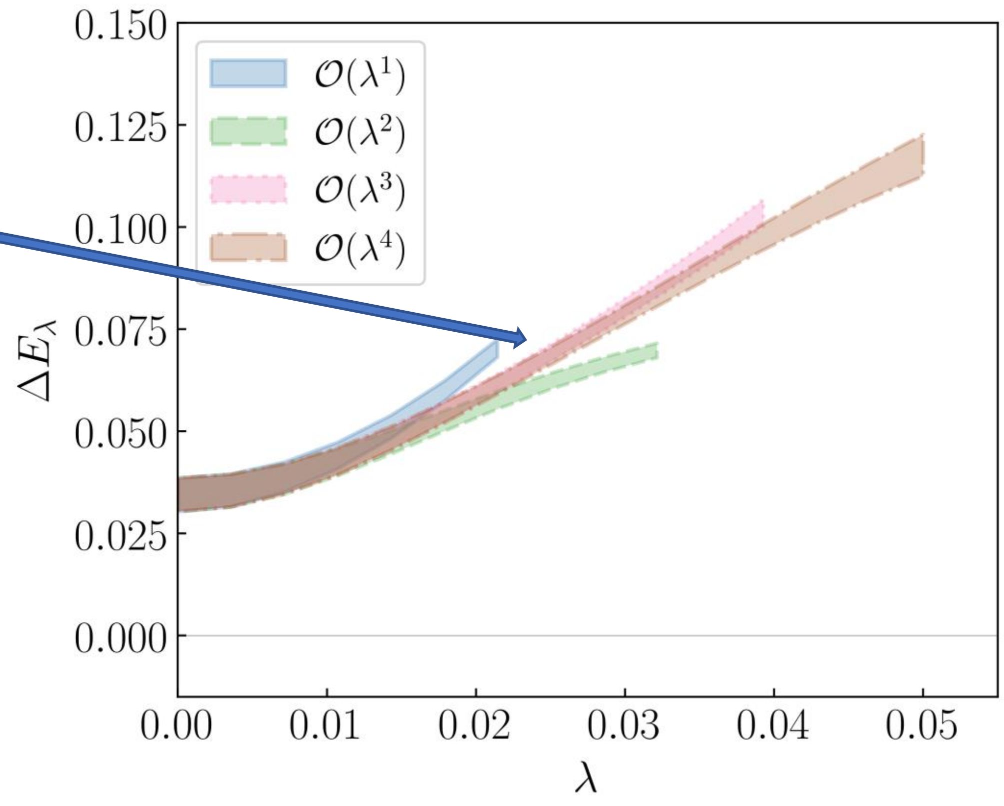
- This cancels out more correlations
- Fit to get the slope in λ
- Relate the slope to the matrix element

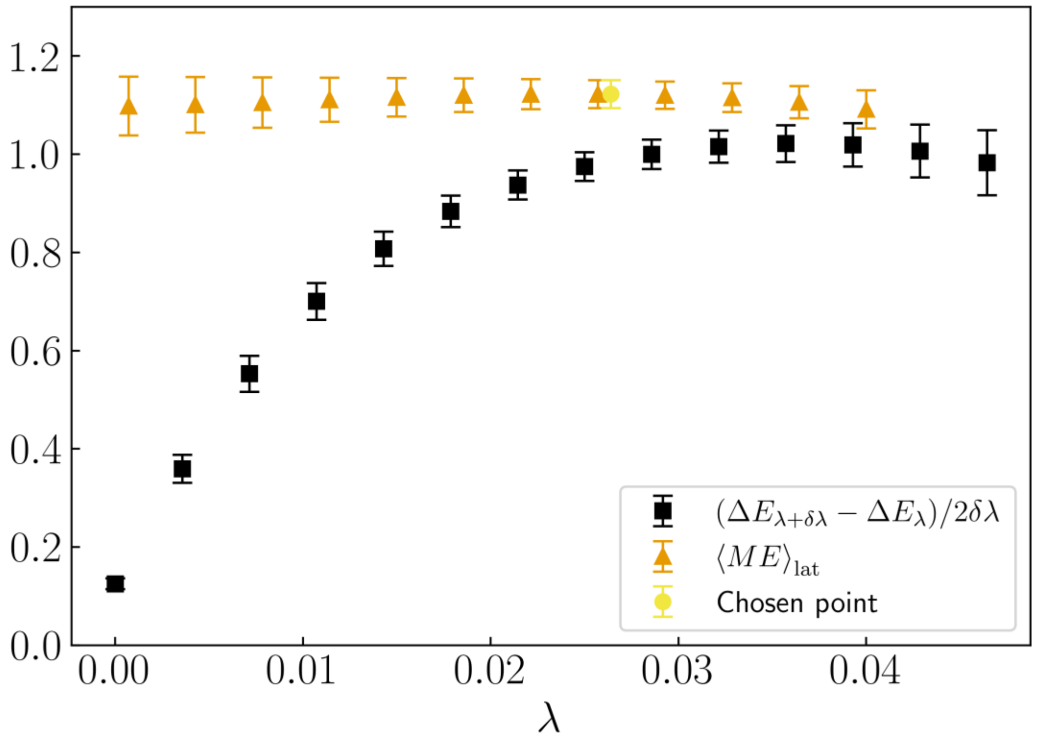


Does the method work at Q^2=0?

The expansion in λ holds for a smaller range:

O(λ^3) and O(λ^4) diverge around λ =0.03





<= Fitting to the slope at small λ still produces stable results

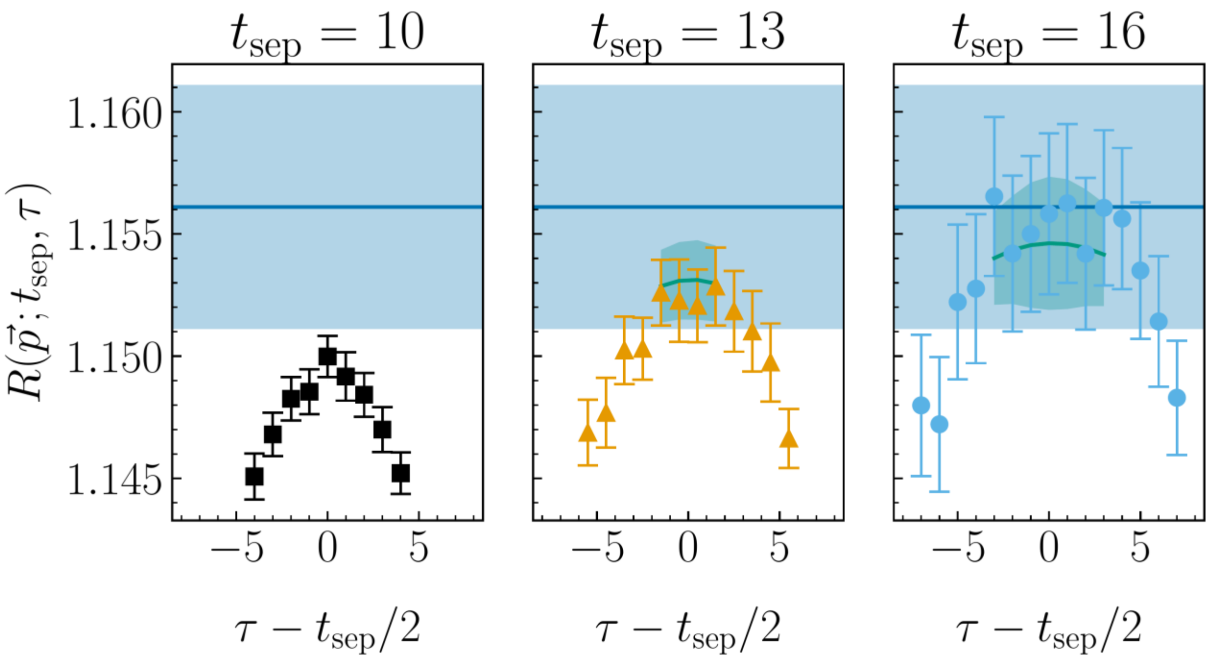
$$\Delta E_{\lambda\Sigma N} = \sqrt{(E_N(\vec{q}) - M_\Sigma)^2 + 4\lambda^2 \left| \langle N(\vec{q})|\bar{u}\gamma_4 s|\Sigma(\vec{0})\rangle \right|^2}$$

How does this compare to three-point function results?

3-point function

- Same # of gauge configurations
- Both $\Sigma^- \rightarrow n$ and opposite 3-point functions used.
- 3 source-sink separations:
    - t=10,13,16 (0.74, 0.96, 1.18 fm)
- Fit ansatz includes all three t_{sep} and the first excited state

Ratio of 3pt and 2pt functions at Q^2=0

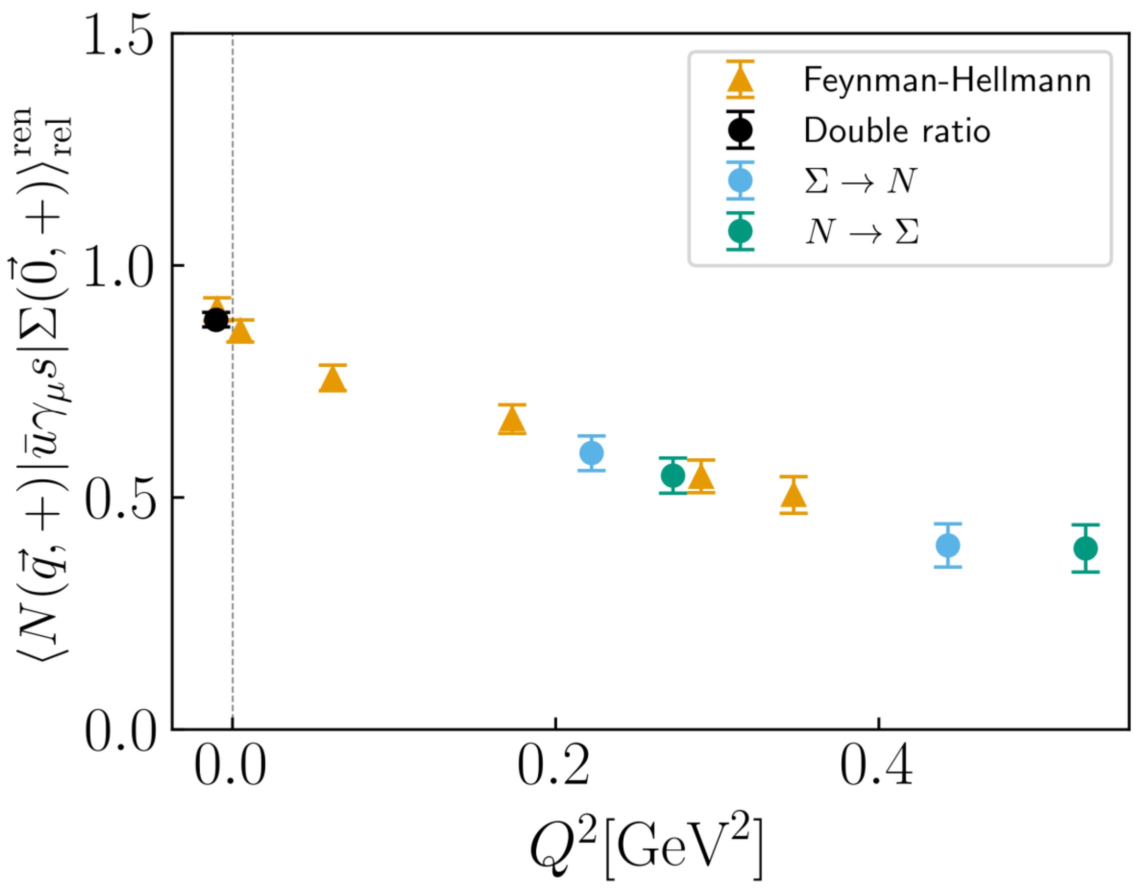


The dependence on t_{sep} has not been eliminated:
=> Has the ground state been saturated?

How does this compare to three-point function results?

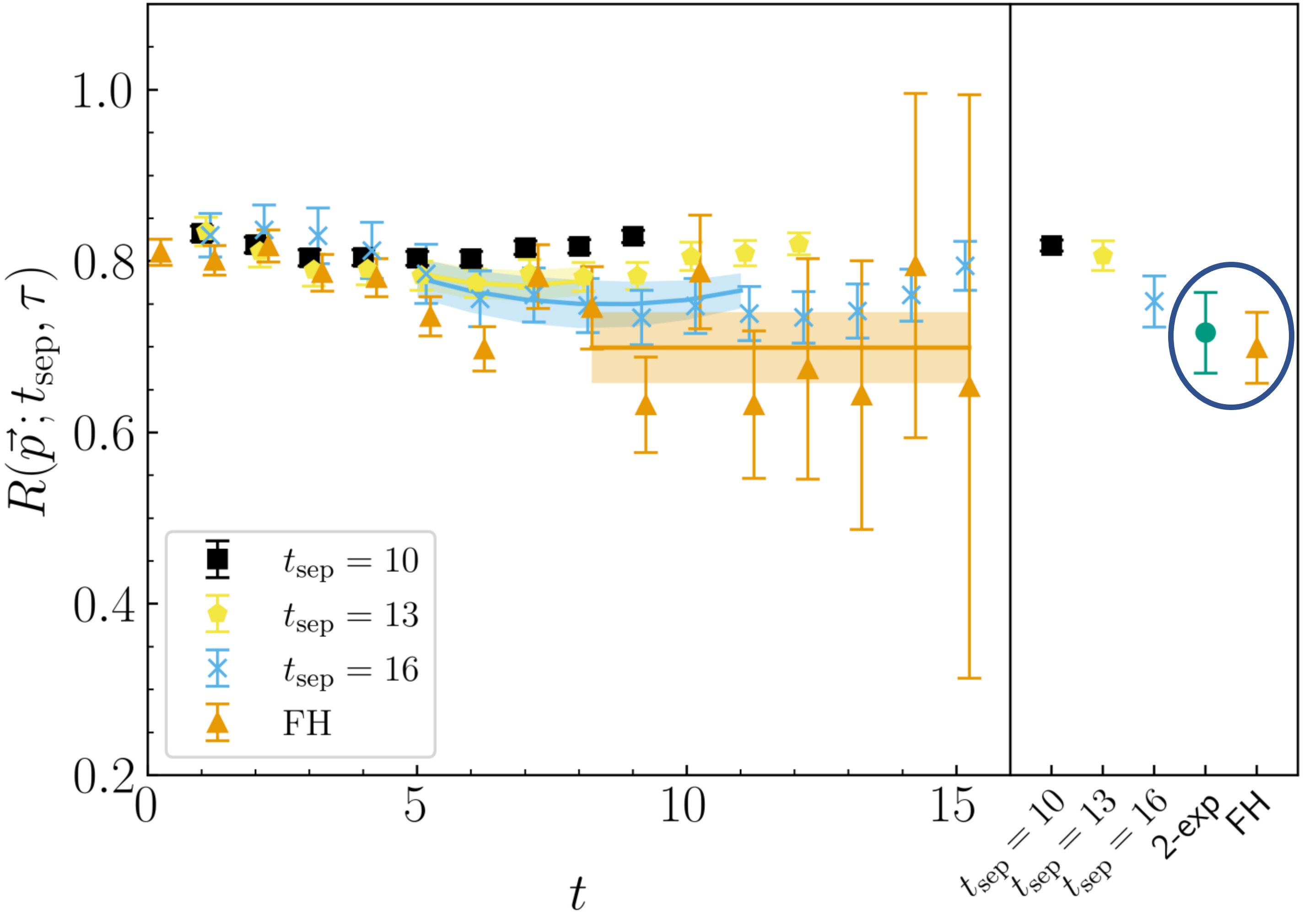
Matrix element as a function of Q^2:

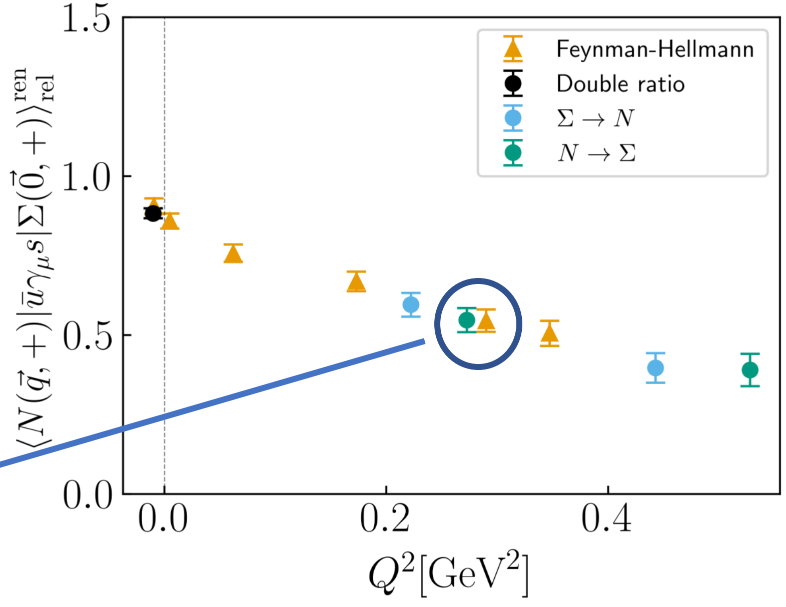
3-point function results at non-zero Q^2 only used one transition direction



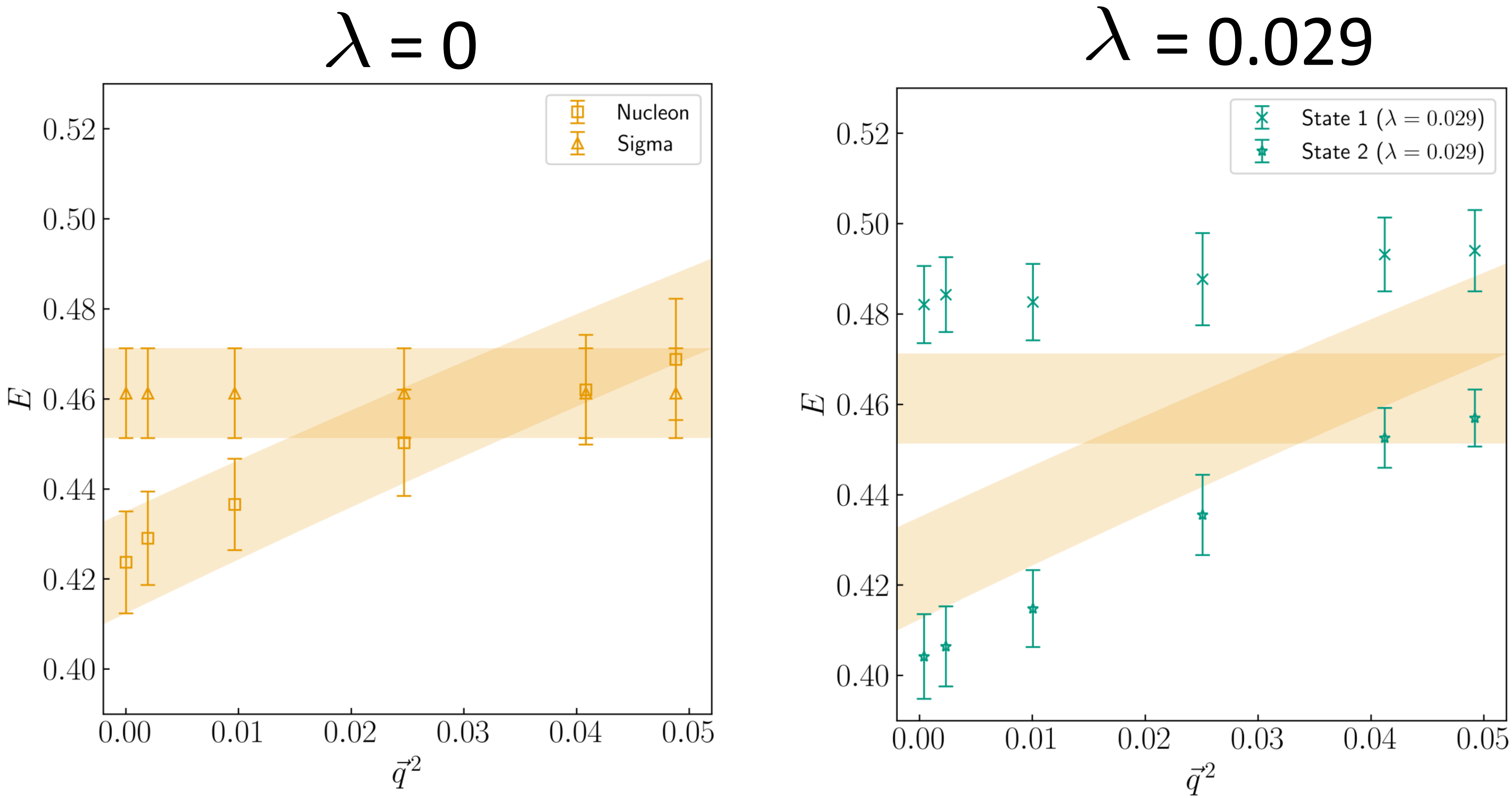
How does this compare to three-point function results?

Compare two Q^2 points more closely:





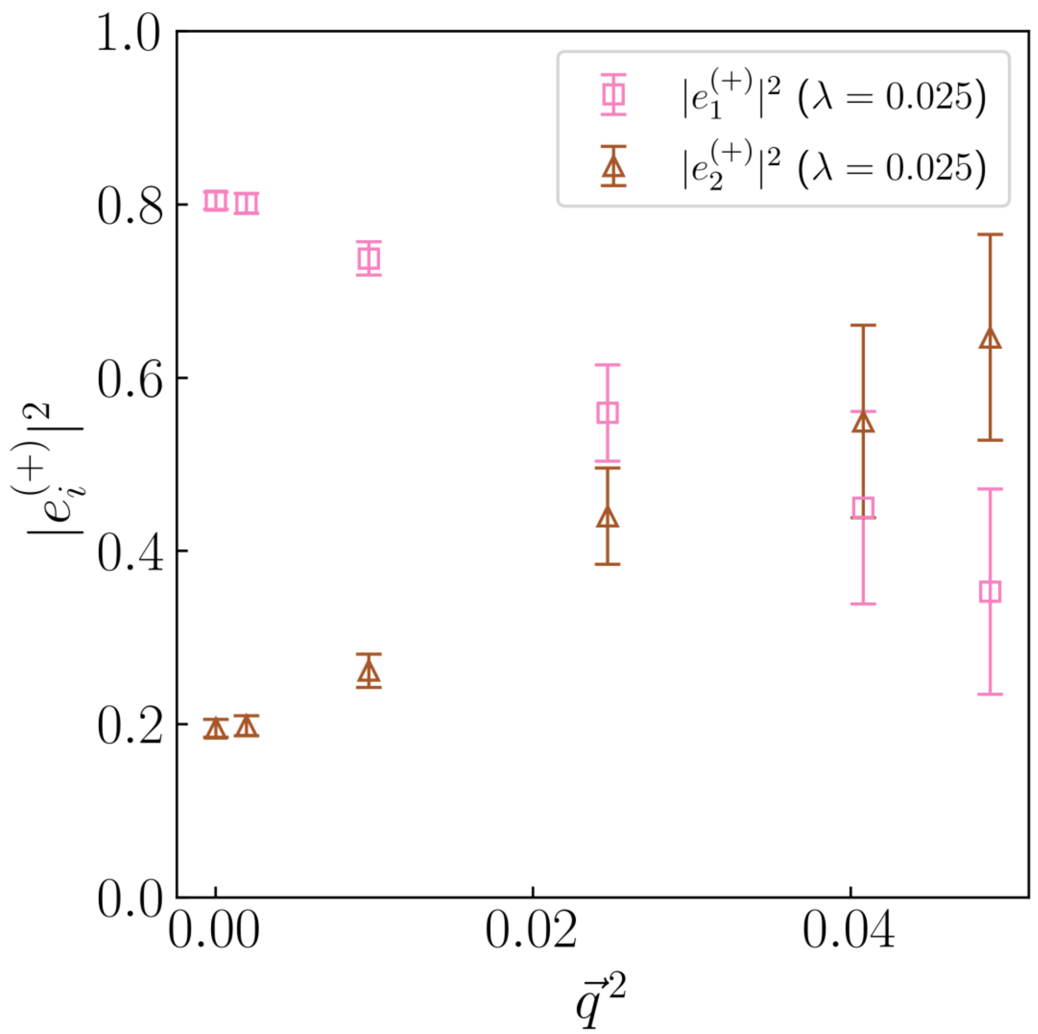
Avoided level crossing



Eigenvectors

- The eigenvectors of the GEVP show the mixing between the states as a function of q^2

$$\vec{e}^{(\pm)} = \begin{pmatrix} e_1^{(\pm)} \\ e_2^{(\pm)} \end{pmatrix}$$



Conclusion

- The Feynman-Hellman method can be used to calculate hyperon transition form factors
- Only requires one Euclidean time parameter to be optimised to extract the ground state.
- Using multiple different operators will allow for the extraction of separate form facors
- Method should be tested on lattices with larger splittings between the light and strange quarks