Factorization connecting continuum and lattice TMDs

Jefferson Lab Theory Seminar

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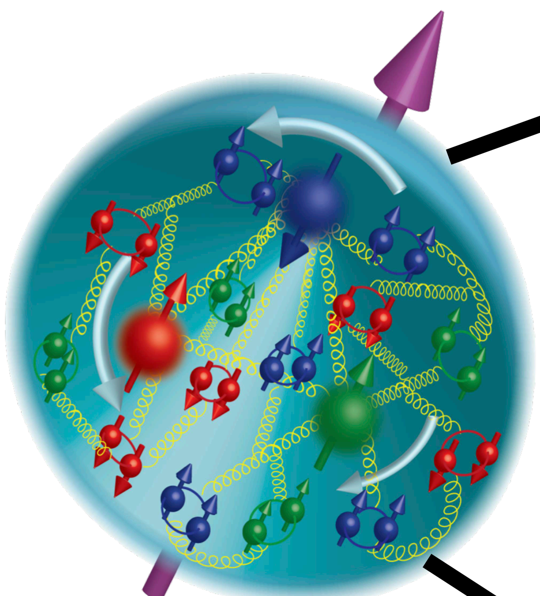
FEB 21, 2022

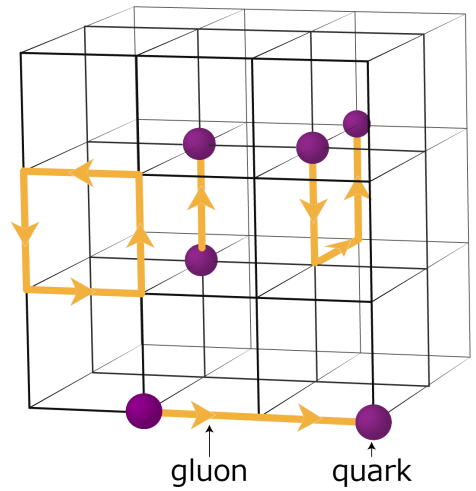
Argonne NATIONAL LABORATORY

In collaboration with M. Ebert, S. Schindler and I. Stewart,
based on arXiv: 2201.08401.

3D Tomography of the Proton (Hadrons)

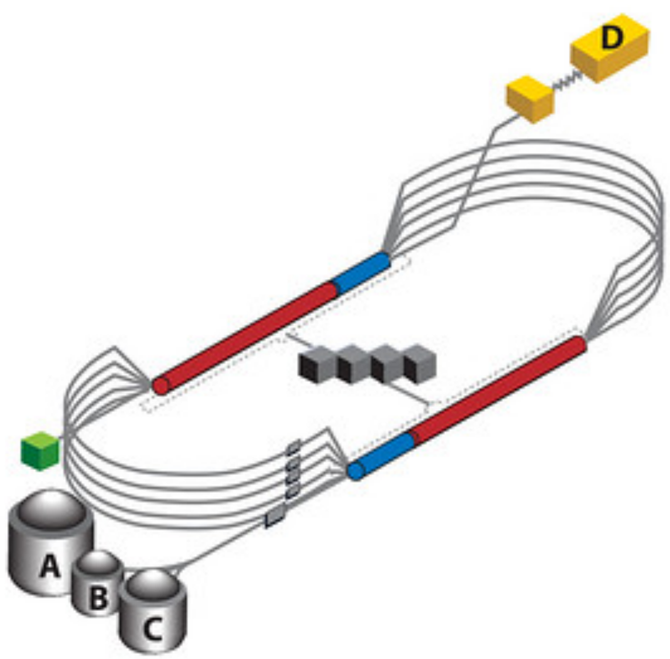


Lattice QCD

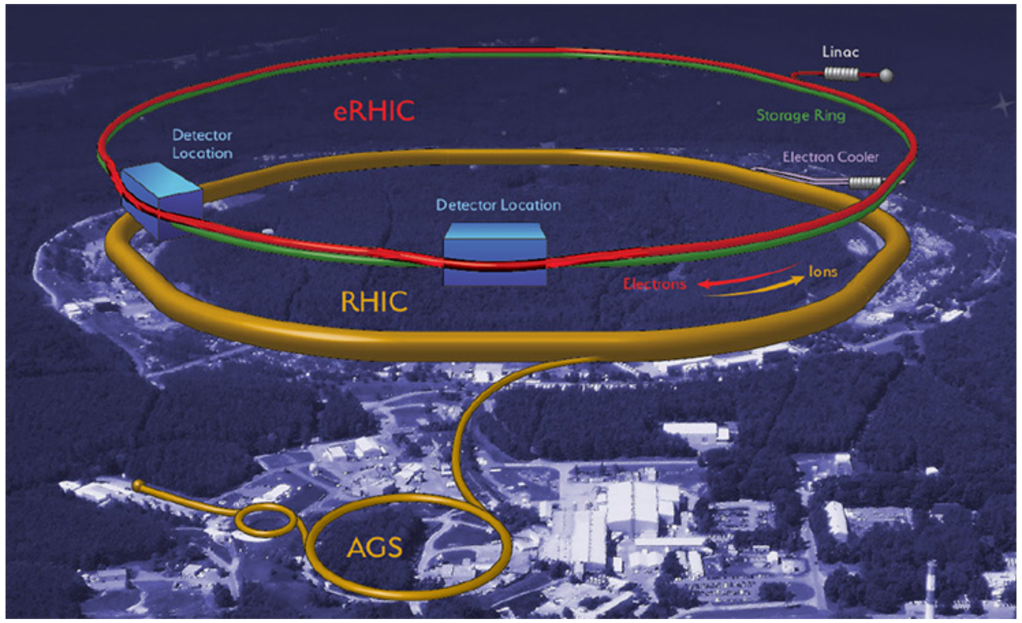


+

Hard Scattering



Jefferson Lab 12 GeV



The Electron-Ion Collider

Outline

- **Introduction to TMDs**
- **Lattice TMDs**
  - LaMET and Quasi-TMDs
  - Lorentz-invariant approach (MHENS scheme)
- **Relation between lattice and continuum TMDs**
- **First lattice results**
  - Collins-Soper kernel for TMD evolution
  - Soft function

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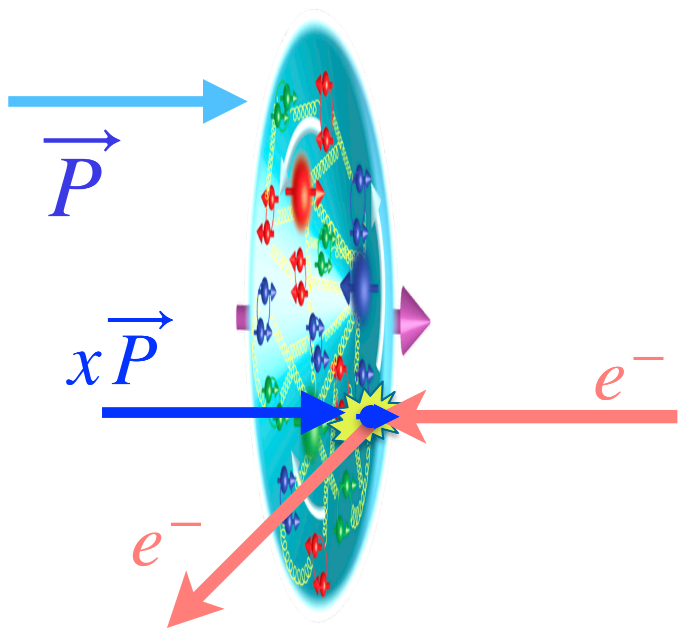
High-energy proton structure and the parton model

The inner proton structure has been probed through high-energy scattering.



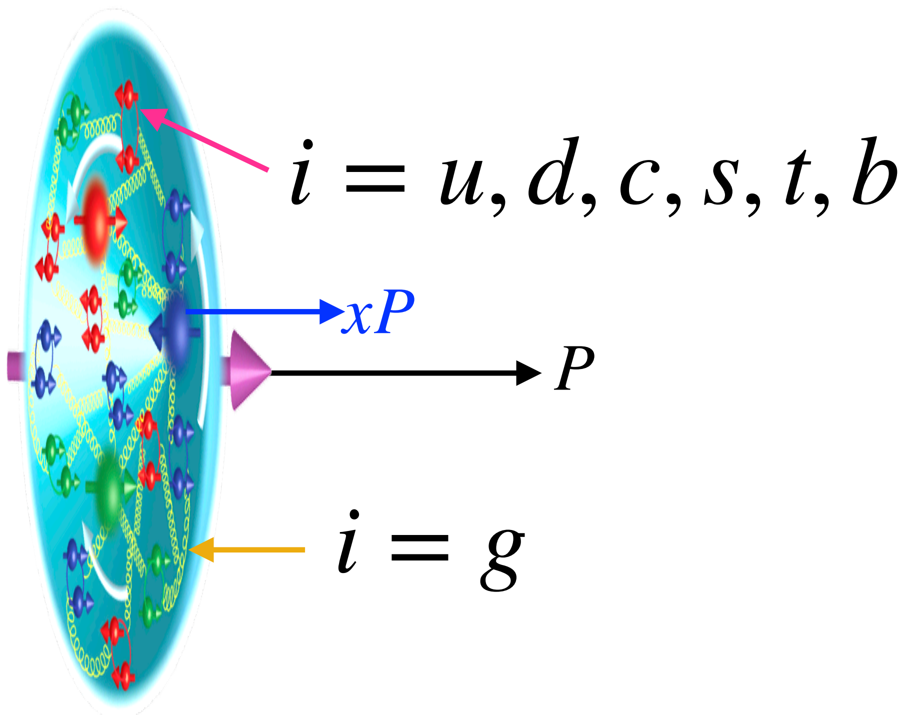
Richard P. Feynman

Feynman's parton model (1969):

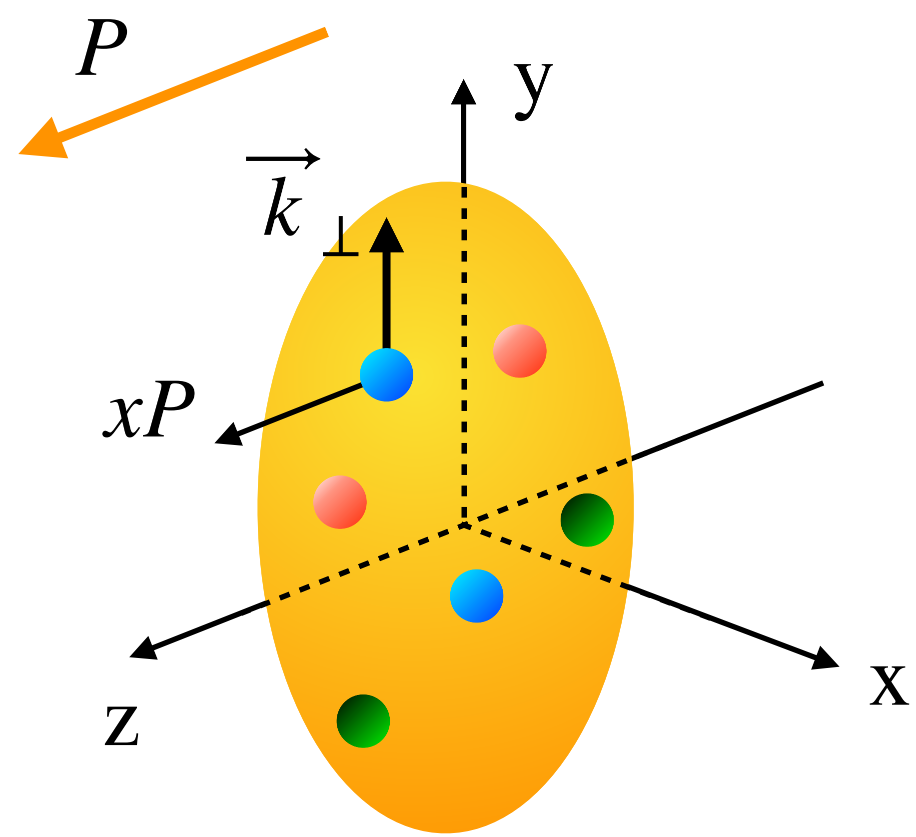
- When the proton travels at almost the speed of light, quarks and gluons are "frozen" in the transverse plane due to Lorentz contraction;
- During a hard collision, the struck quark/gluon (parton) appears to the probe that it does not interact with its surroundings.

Tomography in the 3D momentum space

Collinear parton distribution function (PDF) $f_i(x)$



Transverse-momentum dependent parton distribution (TMD) $f_i(x, \vec{k}_\perp)$

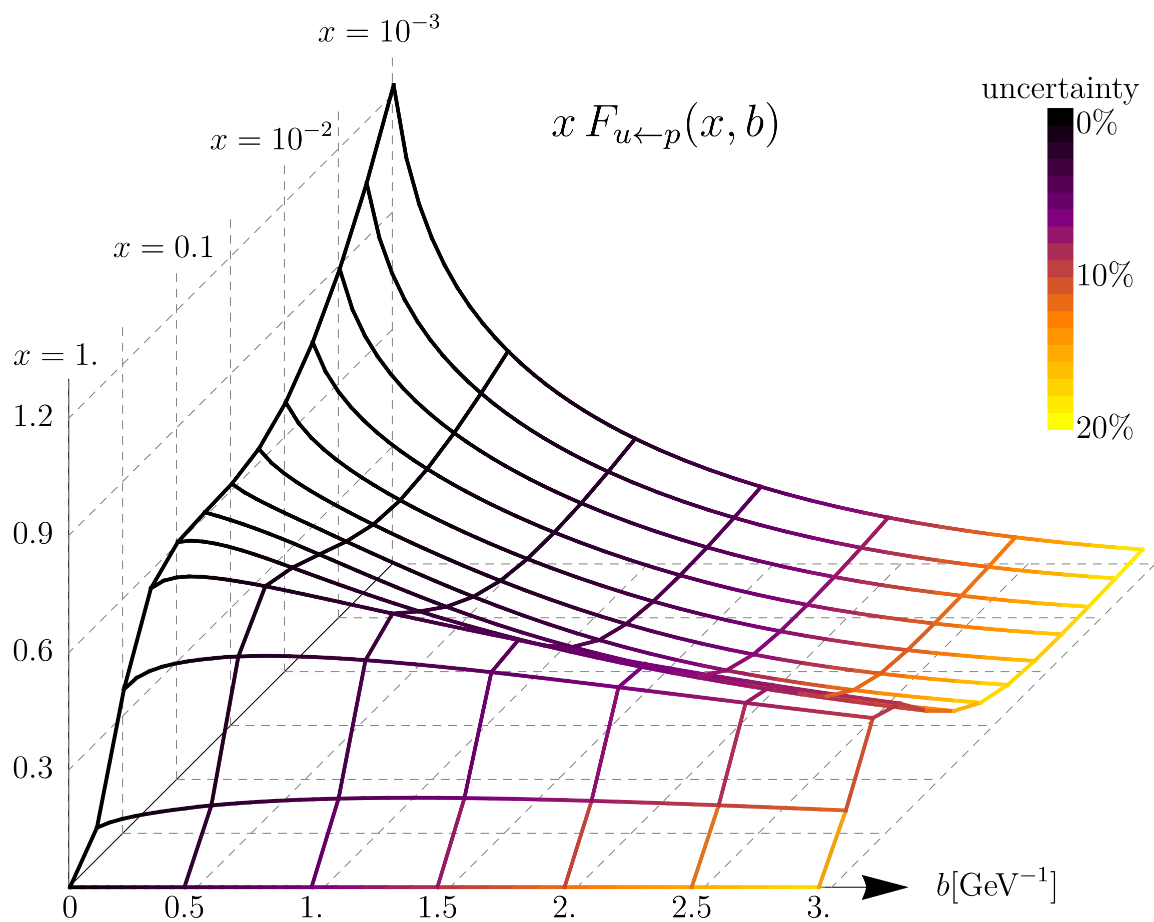


TMD in the Fourier conjugate b_T-space:

$$f_i(x, \vec{b}_T) = \int d^2k_T \, e^{i\vec{k}_T \cdot \vec{b}_T} f_i(x, \vec{k}_T)$$

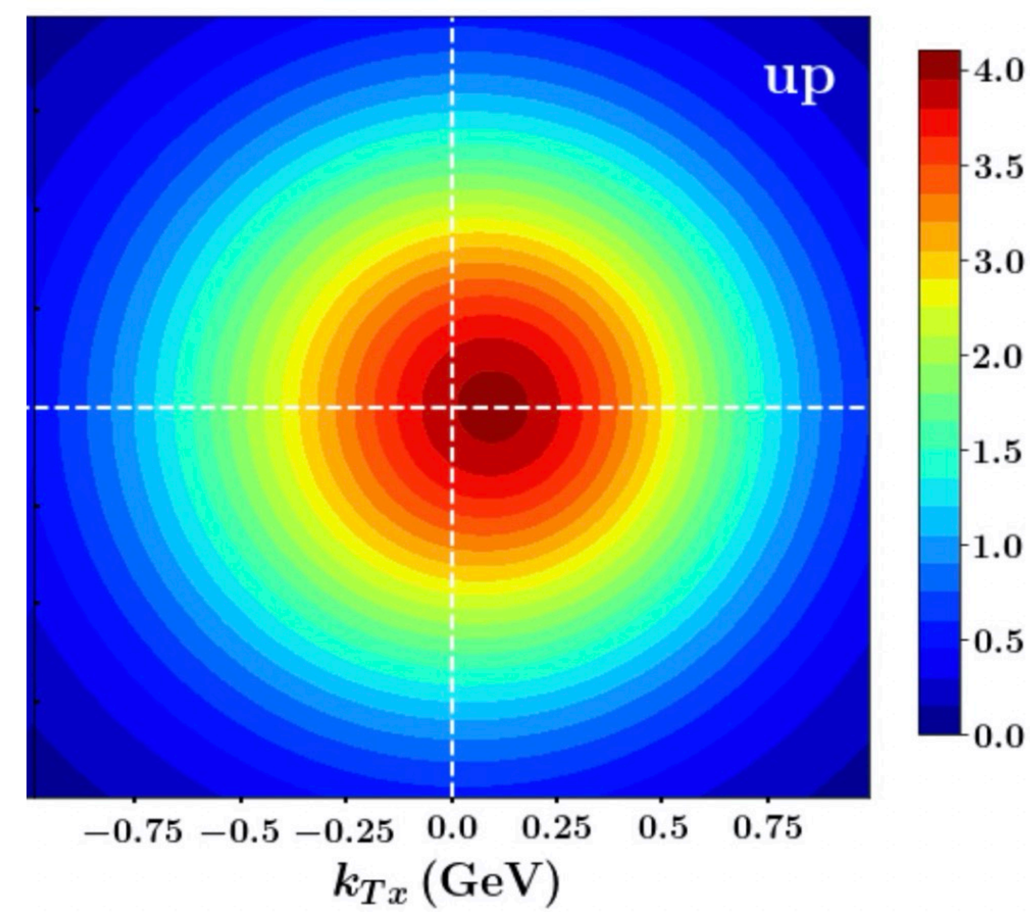
Tomography in the 3D momentum space

Unpolarized quark TMD



I. Scimemi and A. Vladimirov, JHEP **06** (2020).

Quark Sivers function



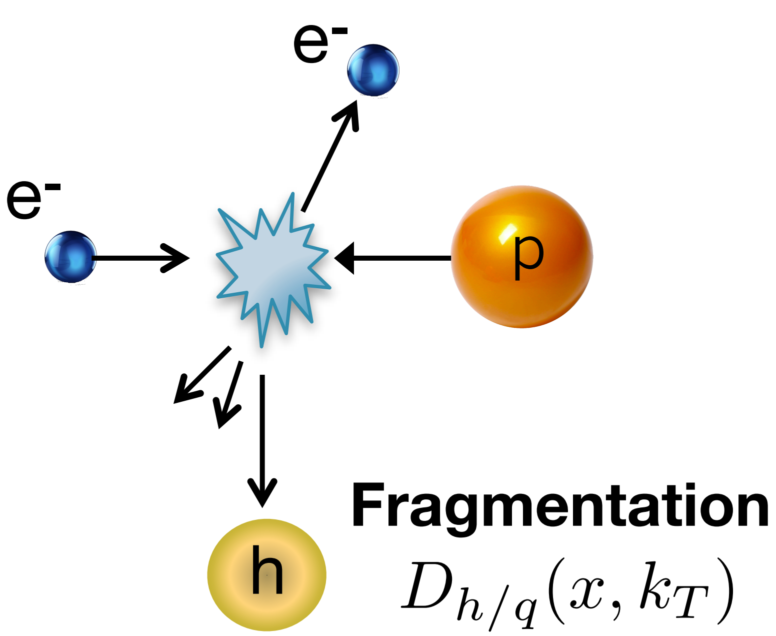
Cammarota, Gamberg, Kang et al. (JAM Collaboration), PRD **102** (2020).

TMD from experiments

TMD processes:

Semi-Inclusive DIS

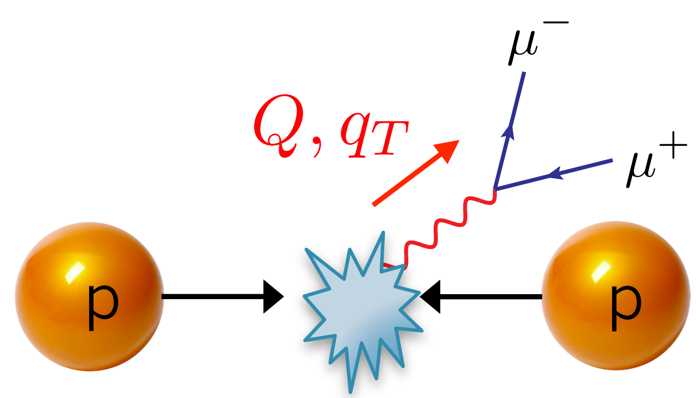
$$\sigma \sim f_{q/P}(x, k_T) D_{h/q}(x, k_T)$$



HERMES, COMPASS, JLab, EIC, …

Drell-Yan

$$\sigma \sim f_{q/P}(x, k_T) f_{q/P}(x, k_T)$$

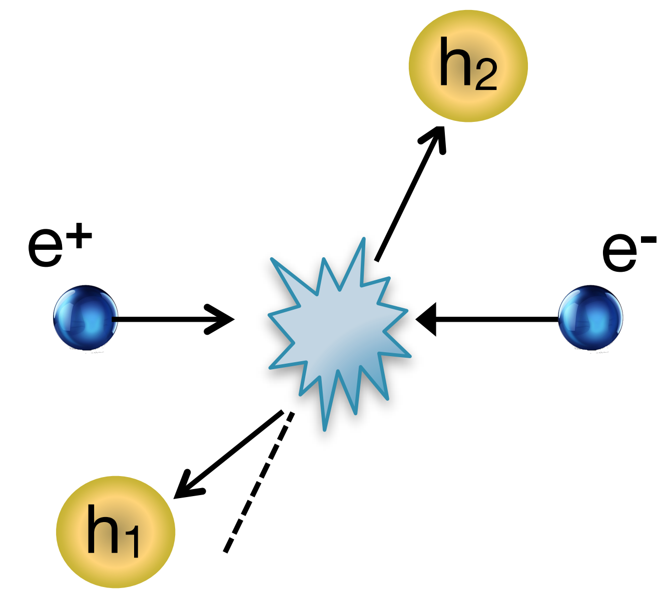


$$q_T \ll Q$$

Fermilab, RHIC, LHC, …

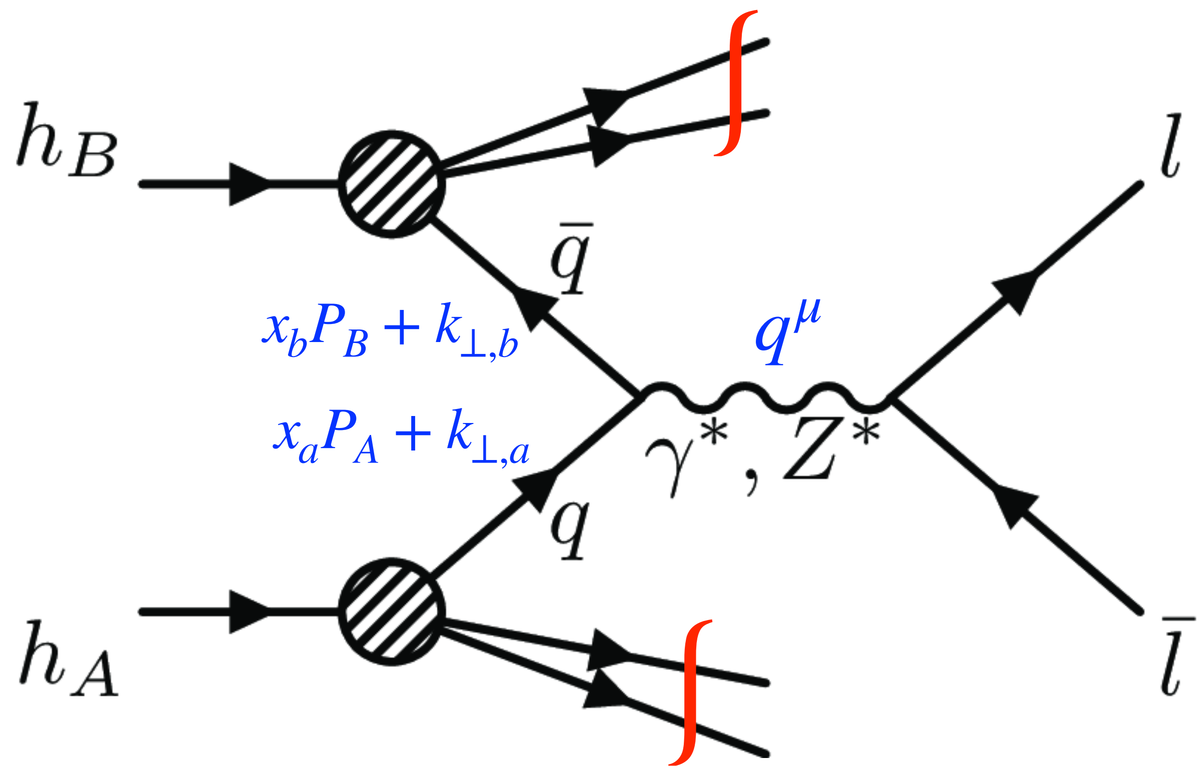
Dihadron in e^+e^-

$$\sigma \sim D_{h_1/q}(x, k_T) D_{h_2/q}(x, k_T)$$



Babar, Belle, BESIII, …

TMD factorization for Drell-Yan processes



$$n_a = (1,0,0,1)/\sqrt{2}\,, \qquad n_b = (1,0,0,-1)/\sqrt{2}$$

$$q^\mu = (q^+, q^-, q_\perp)\,, \quad Q^2 = q^2\,, \quad Y = \frac{1}{2}\ln\frac{n_a\cdot q}{n_b\cdot q}$$

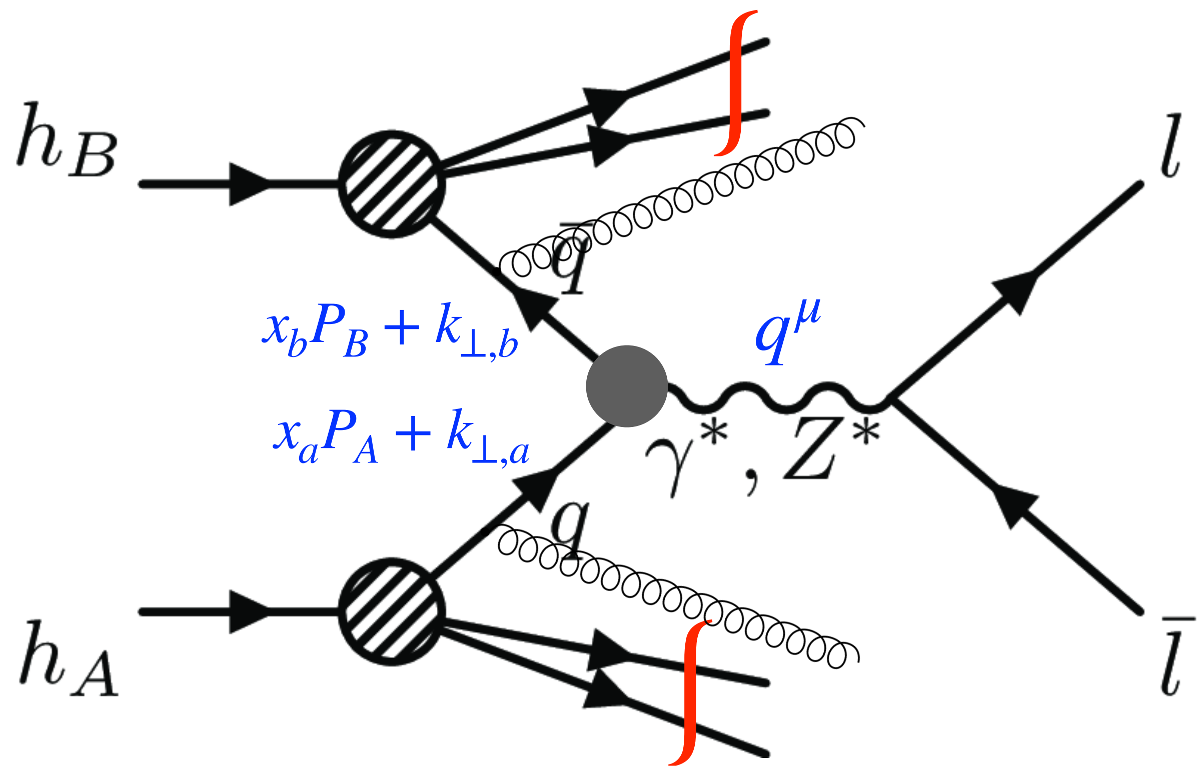
$$s = (P_A + P_B)^2\,, \quad q_T = |q_\perp|$$

$$x_a = Qe^{+Y}\sqrt{s}, \quad x_b = Qe^{-Y}\sqrt{s}, \quad Q, s \gg \Lambda_{\rm QCD}$$

Differential cross section $\frac{d\sigma_{\rm DY}}{dQdYd^2q_T}$:

- Q and Y uniquely determine x_a and x_b of initial state partons;
- q_T is contributed from the transverse momenta of partons and radiated gluons.

TMD factorization for Drell-Yan processes



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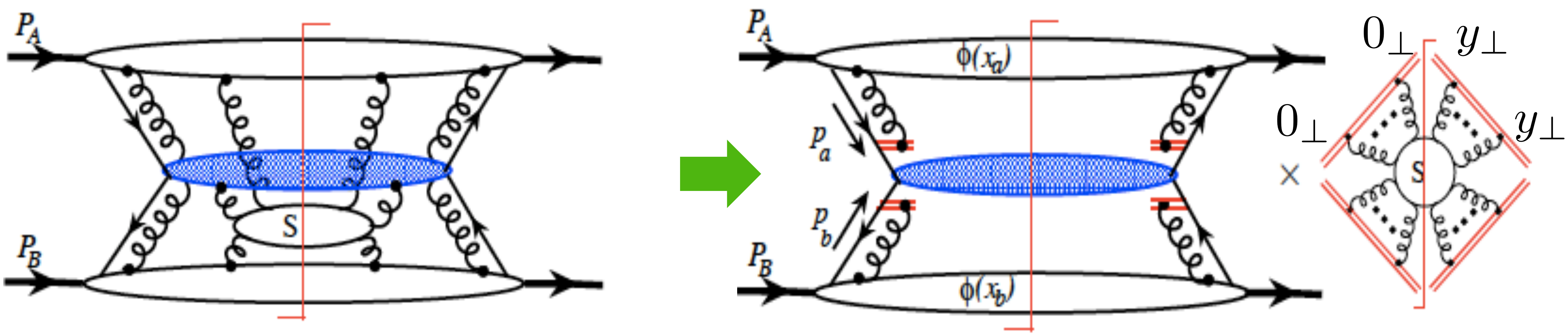
TMD factorization for Drell-Yan processes

TMD factorization for $q_T = |q_\perp| \ll Q$:



From J. Qiu's TMD School 2022 Lectures.

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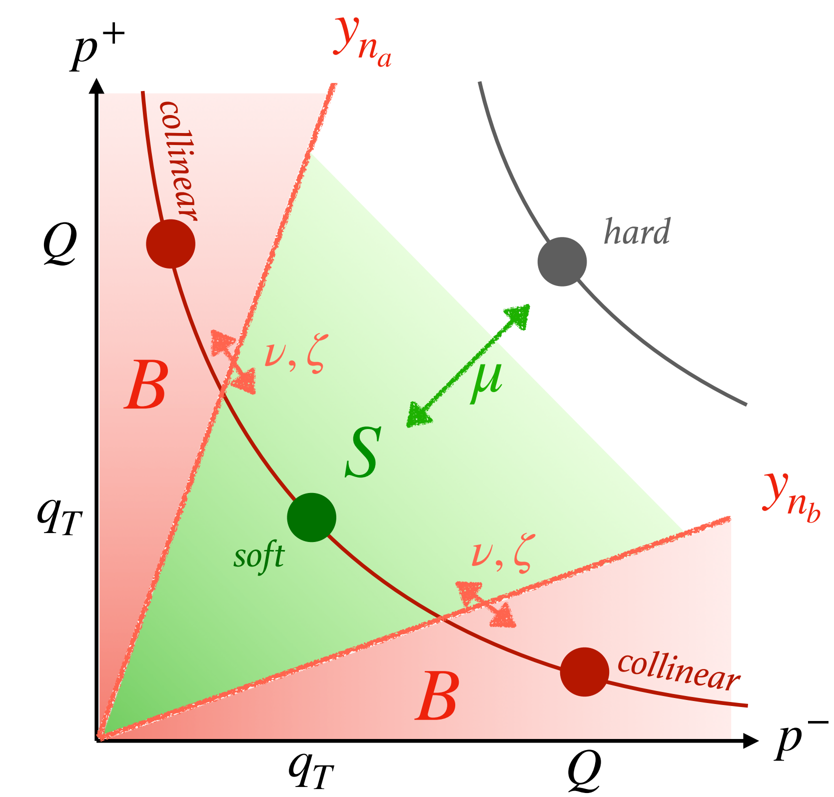

TMD Handbook, by TMD Collaboration

(Schematic) factorization formula:

$$\frac{d\sigma_{\mathrm{DY}}}{dQdYd^2q_T} = H \otimes B \otimes B \otimes S$$

Hard factor | **(Collineear) beam functions** | **Soft function**

TMD factorization for Drell-Yan processes

TMD factorization for $q_T = |q_\perp| \ll Q$:

Separation of soft (p_s) and collinear (p_{n_a}, p_{n_b}) modes:

$$p_{n_a} \sim Q(\lambda^2, 1, \lambda)\,,\ p_{n_b} \sim Q(1, \lambda^2, \lambda)\,,\ p_s \sim Q(\lambda, \lambda, \lambda)\,,\ \lambda \ll 1$$

$$p_{n_a}^2 \sim p_{n_b}^2 \sim p_s^2 \sim q_T^2\,, \qquad \text{rapidity}: \quad y = \frac{1}{2}\ln\frac{p^-}{p^+}$$

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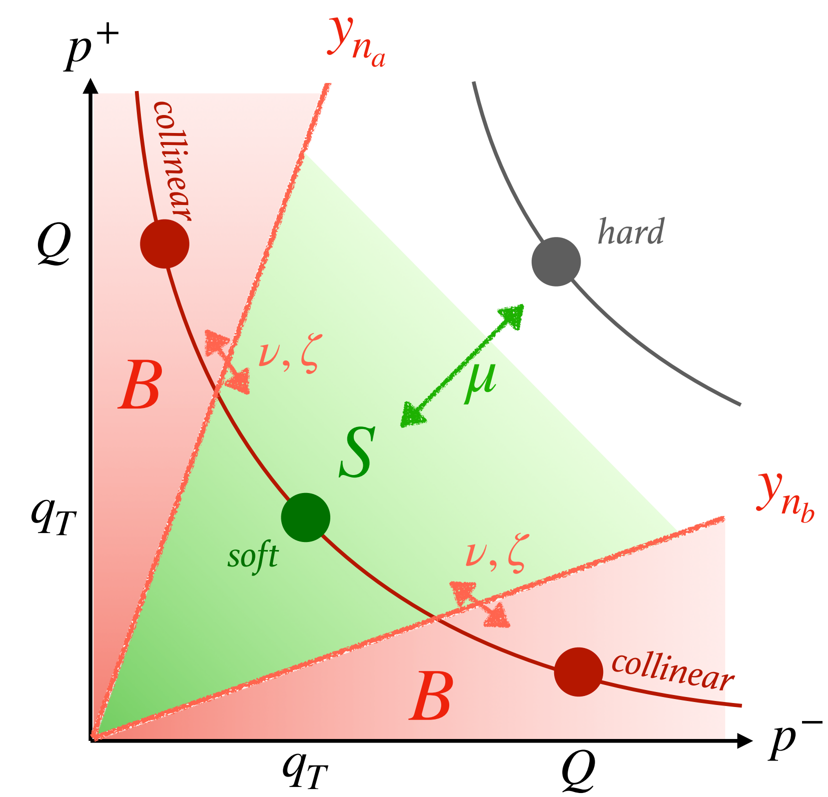
Hard factor — **(Collinear) beam functions** — **Soft function**



TMD Handbook, by TMD Collaboration

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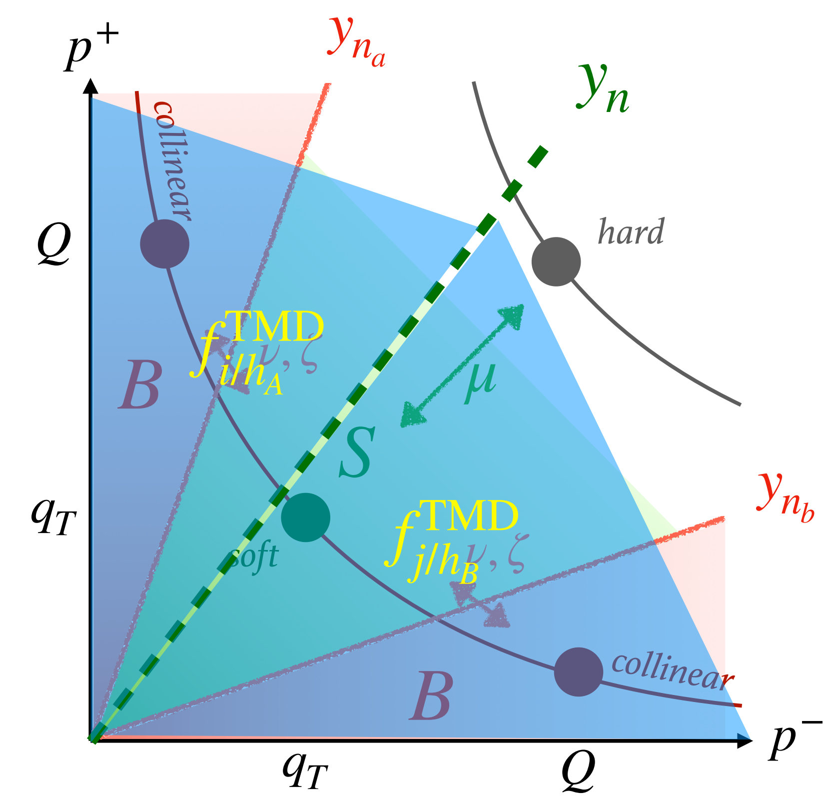

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(Schematic) factorization formula:

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Hard factor **(Collinear) beam functions** **Soft function**

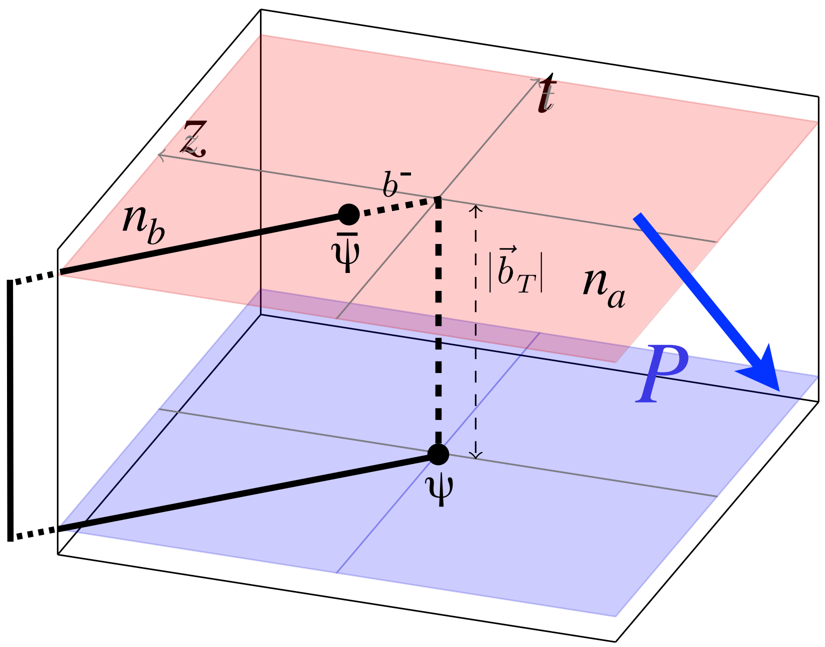
$$= H \otimes f^{\text{TMD}} \otimes f^{\text{TMD}}$$

Physical TMDs

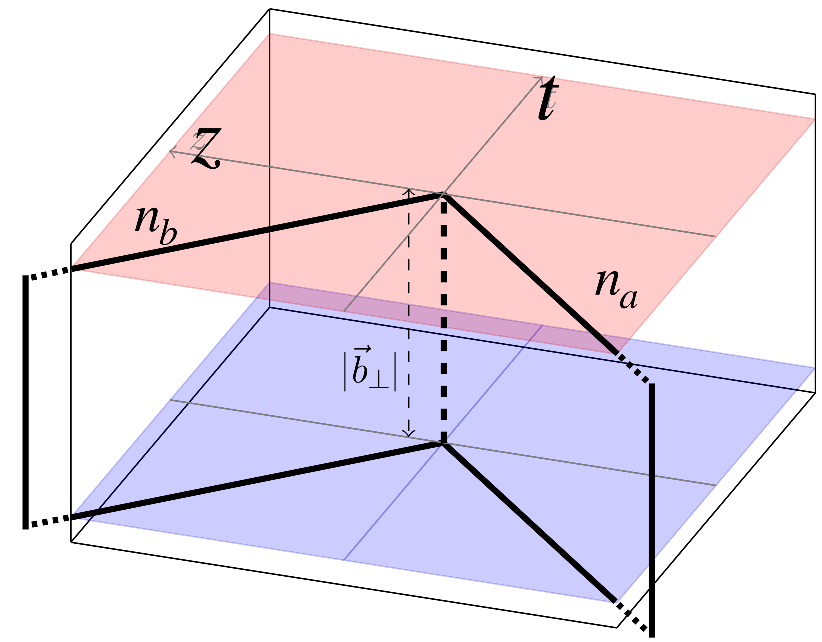
$$f^{\text{TMD}} = B\sqrt{S}$$

TMD definition

- Beam function:
- Soft function :



Hadronic matrix element



Vacuum matrix element

Rapidity divergence regulator

$$f_i^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{\mathrm{UV}}(\epsilon, \mu, xP^+) \lim_{\tau \to 0} B_i(x, \vec{b}_T, \epsilon, \tau, xP^+) \Delta_S^i(b_T, \epsilon, \tau)$$

UV divergence regulator

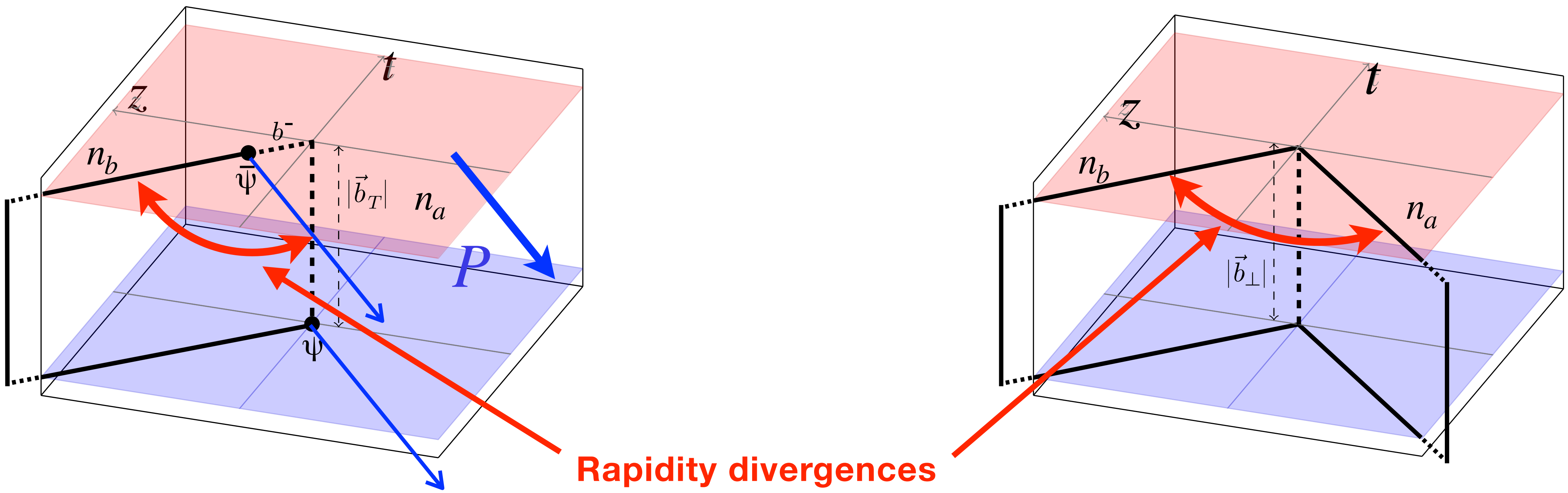
Soft factor

Collins-Soper scale: $\zeta = 2(xP^+ e^{-y_n})^2$

Ebert, Stewart and YZ, JHEP 09 (2019)

TMD definition

- Beam function:
- Soft function :



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TMD definition

Many different schemes for regulating rapidity divergences:

Wilson lines on the light-cone (SCET):

- Becher and Neubert, EPJC71 (2011);
- Echevarria, Idilbi and Scimemi, JHEP07 (2012), PLB726 (2013);
- Chiu, Jain, Neil and Rothstein, JHEP05 (2012), PRL108 (2012);
- Li, Neil and Zhu, NPB 960 (2020);
- Ebert, Moult, Stewart, Tackman and Vita, JHEP 04 (2019).

Wilson lines off the light-cone:

- **"Collins scheme"**, Collins, Soper and Sterman, NPB250 (1985); Collins, 2011 book;
- **"JMY scheme"**, Ji, Ma and Yuan, PRD71 (2005).

For reviews see also

- Ebert, Stewart and **YZ**, JHEP 09 (2019);
- TMD Handbook, by TMD collaboration.

Scheme-independent TMD factorization:

$$\frac{d\sigma_{\rm DY}}{dQdYd^2q_T} = \sigma_0 \sum_{i,j} H_{ij}(Q,\mu) \int d^2b_T\, e^{i\vec{b}_T\cdot\vec{q}_T} f_i^{\rm TMD}(x_a, \vec{b}_T, \mu, \zeta_a) f_j^{\rm TMD}(x_b, \vec{b}_T, \mu, \zeta_b) \Big[1 + \mathcal{O}\Big(\frac{q_T^2}{Q^2}, \frac{\Lambda_{\rm QCD}^2}{Q^2}\Big)\Big]$$

$$\zeta_a \zeta_b = Q^4$$

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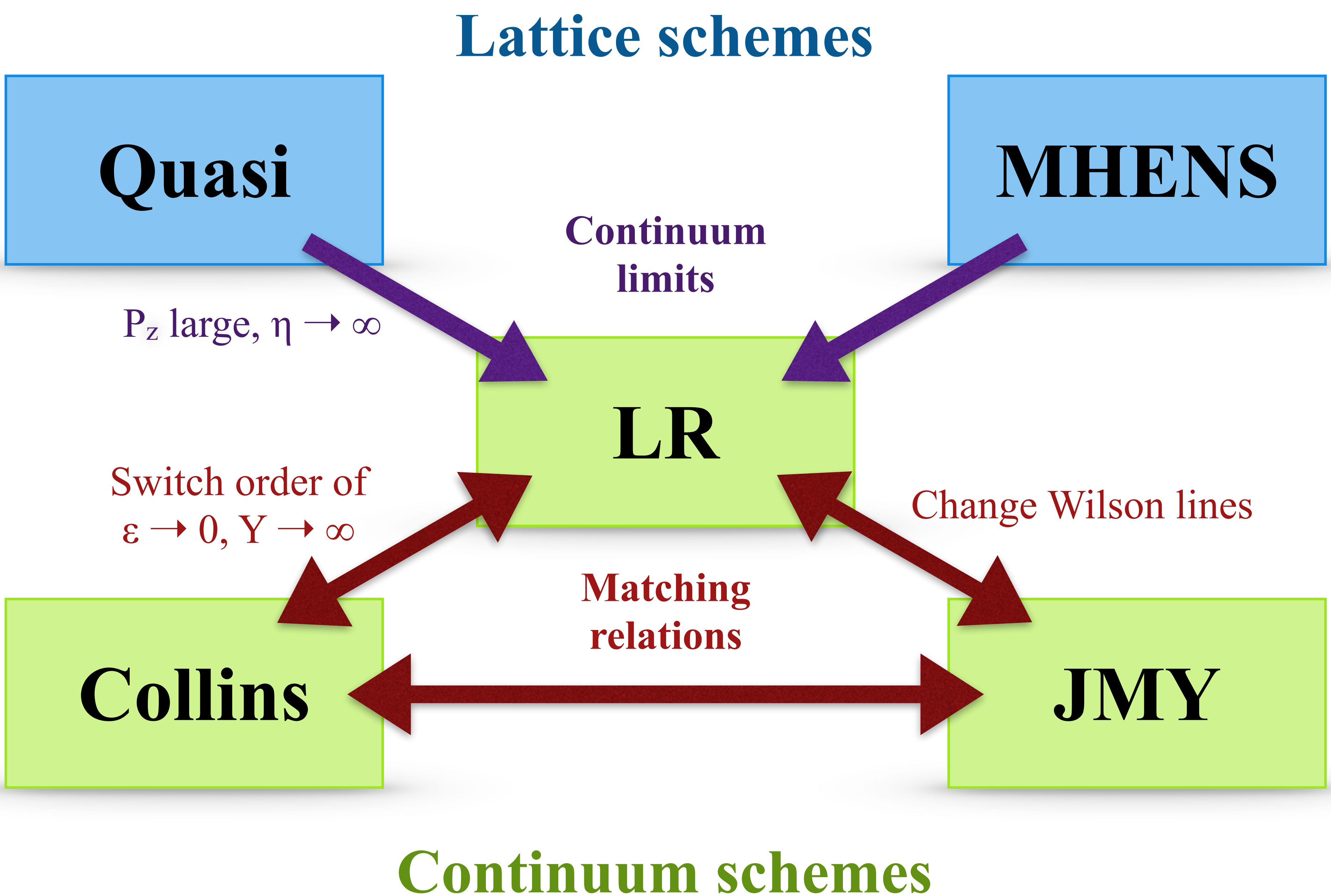
$$\zeta_a\zeta_b = Q^4$$

- For $b_T \ll \Lambda_{\rm QCD}^{-1}$, can be perturbatively matched onto collinear PDFs;
- For $b_T \sim \Lambda_{\rm QCD}^{-1}$, becomes intrinsically non-perturbative, which motivates first-principles calculations.

Overview of the continuum and lattice schemes

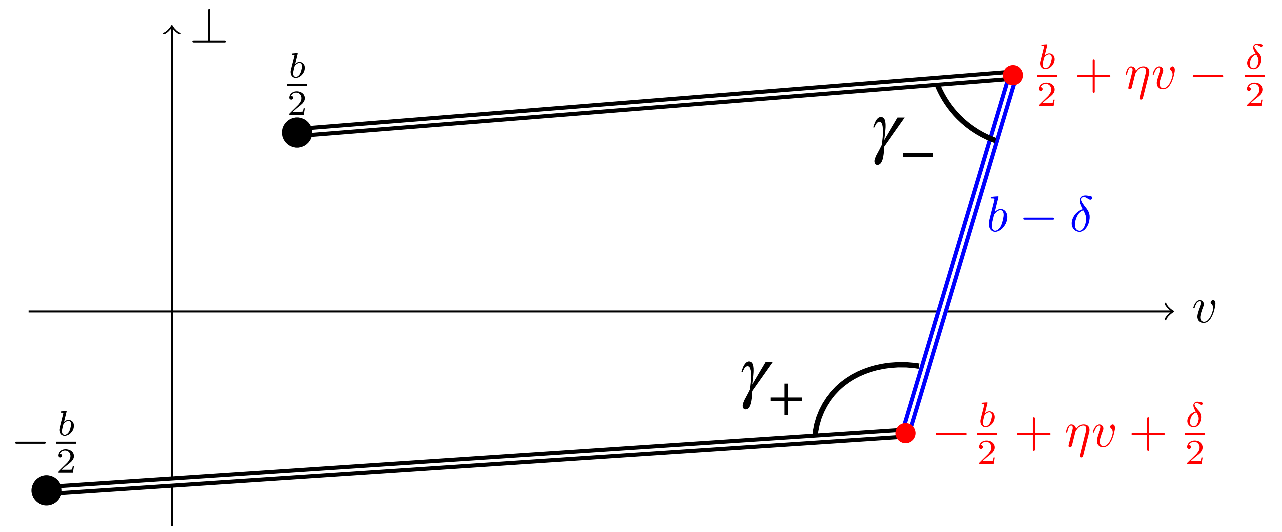
Ebert, Schindler, Stewart and YZ, 2201.08401.



General correlator for the beam function

$$\Omega^{[\Gamma]}_{q_i/h}(b, P, \epsilon, \eta v, \delta) = \left\langle h(P) \middle| \bar{q}_i\left(\frac{b}{2}\right)\frac{\Gamma}{2} W^F_{\sqsupset}(b, \eta v, \delta) q_i\left(-\frac{b}{2}\right) \middle| h(P) \right\rangle$$

$$W^R_{\sqsupset}(b, \eta v, \delta) =$$



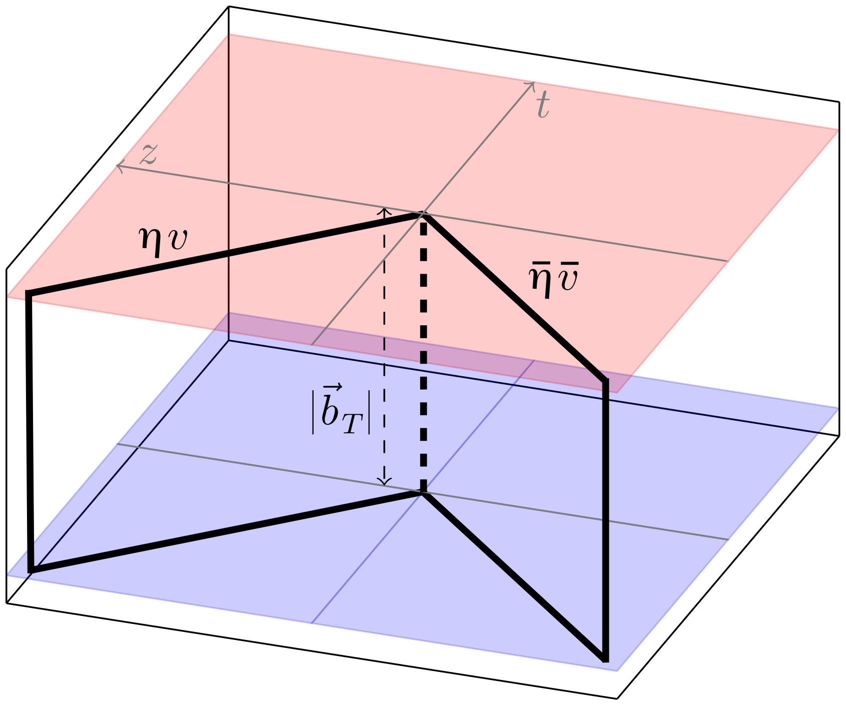
Cusp angles:

$$\cosh \gamma_\pm = \frac{(\eta v \pm \delta/2) \cdot (b - \delta)}{|\eta v \pm \delta/2||b - \delta|}$$

General correlator for the soft function

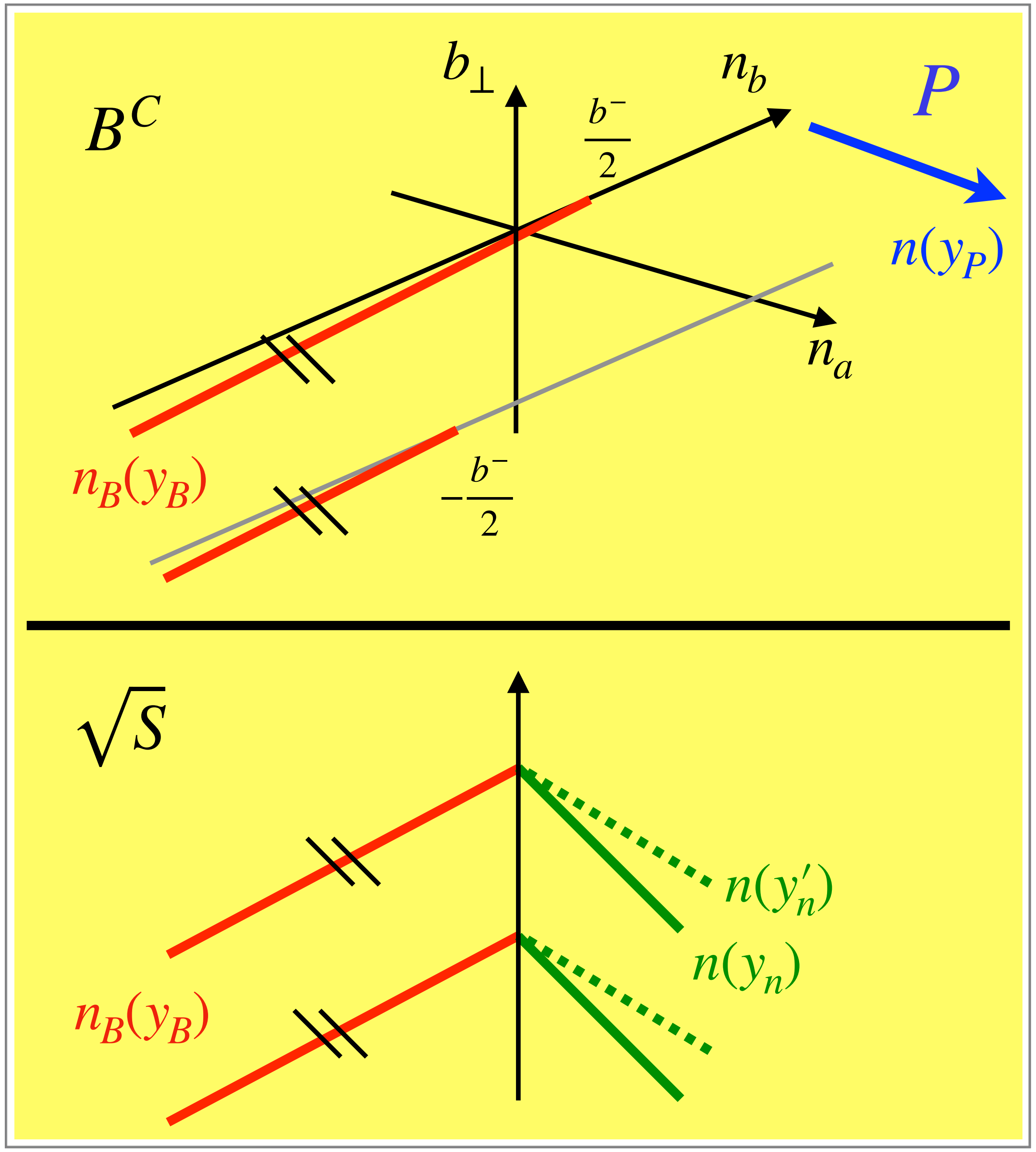
$$S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v}) = \frac{1}{d_R}\left\langle 0 \middle| \mathrm{Tr}\left[S^R_{\gtrless}(b, \eta v, \bar{\eta}\bar{v})\right] \middle| 0 \right\rangle$$

$$S^R_{\gtrless}(b, \eta v, \bar{\eta}\bar{v}) =$$



Collins scheme

$b = (0, b^-, b_\perp)$ $\delta = b^- n_b$



$n^\mu(y_n) \equiv (1, -e^{-2y_n}, 0_\perp)$

$$\delta^\mu = (0, b^-, 0_\perp)$$

$$v^\mu = n_B^\mu(y_B) \equiv n_b^\mu - e^{2y_B} n_a^\mu = (-e^{2y_B}, 1, 0_\perp)$$

$$B^C_{q/h}(x, \vec{b}_T, \epsilon, y_P - y_B) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \times \Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B, b^- n_b\right]$$

$$f^C_{i/h}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{\rm UV}(\epsilon, \mu, \zeta) \times \lim_{y_B \to -\infty} \frac{B^C_{i/h}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}}$$

Rapidity divergences $\propto e^{-\gamma_\zeta^q(b_T,\epsilon)(y_P - y_B)}$, $e^{-\gamma_\zeta^q(b_T,\epsilon)(2y_n - 2y_B)}$

$$\zeta = 2(xP^+ e^{-y_n})^2 = x^2 m_h^2 e^{2(y_P - y_n)}$$

Collins scheme

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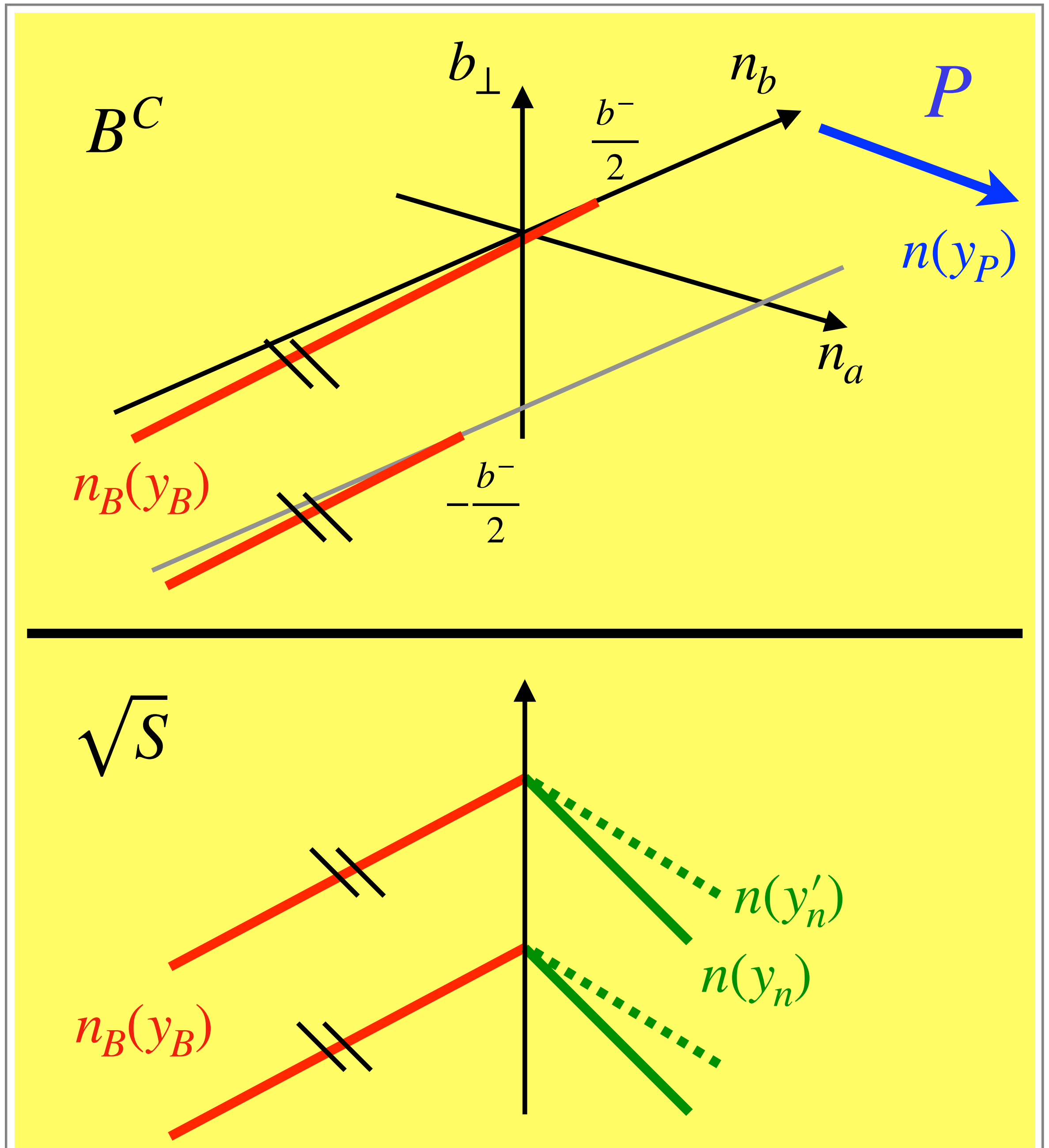

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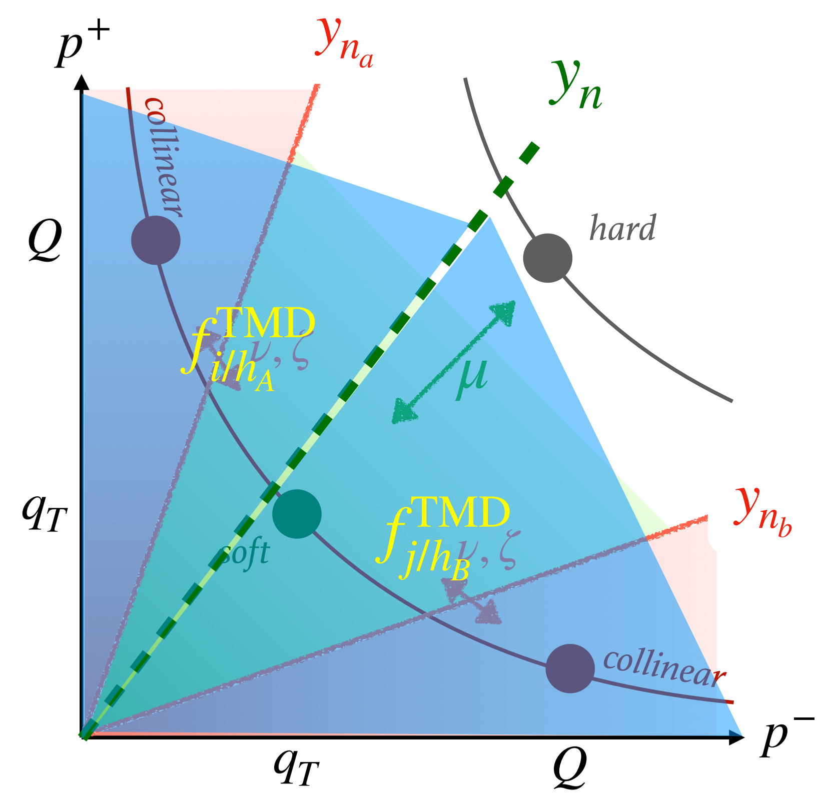


$^{2y_B}, 1, 0_\perp)$

$\supset n_B,\ b^- n_b\Big]$

$$f^C_{i/h}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\rm UV}(\epsilon, \mu, \zeta)$$

$$\times \lim_{y_B\to-\infty} \frac{B^C_{i/h}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}}$$

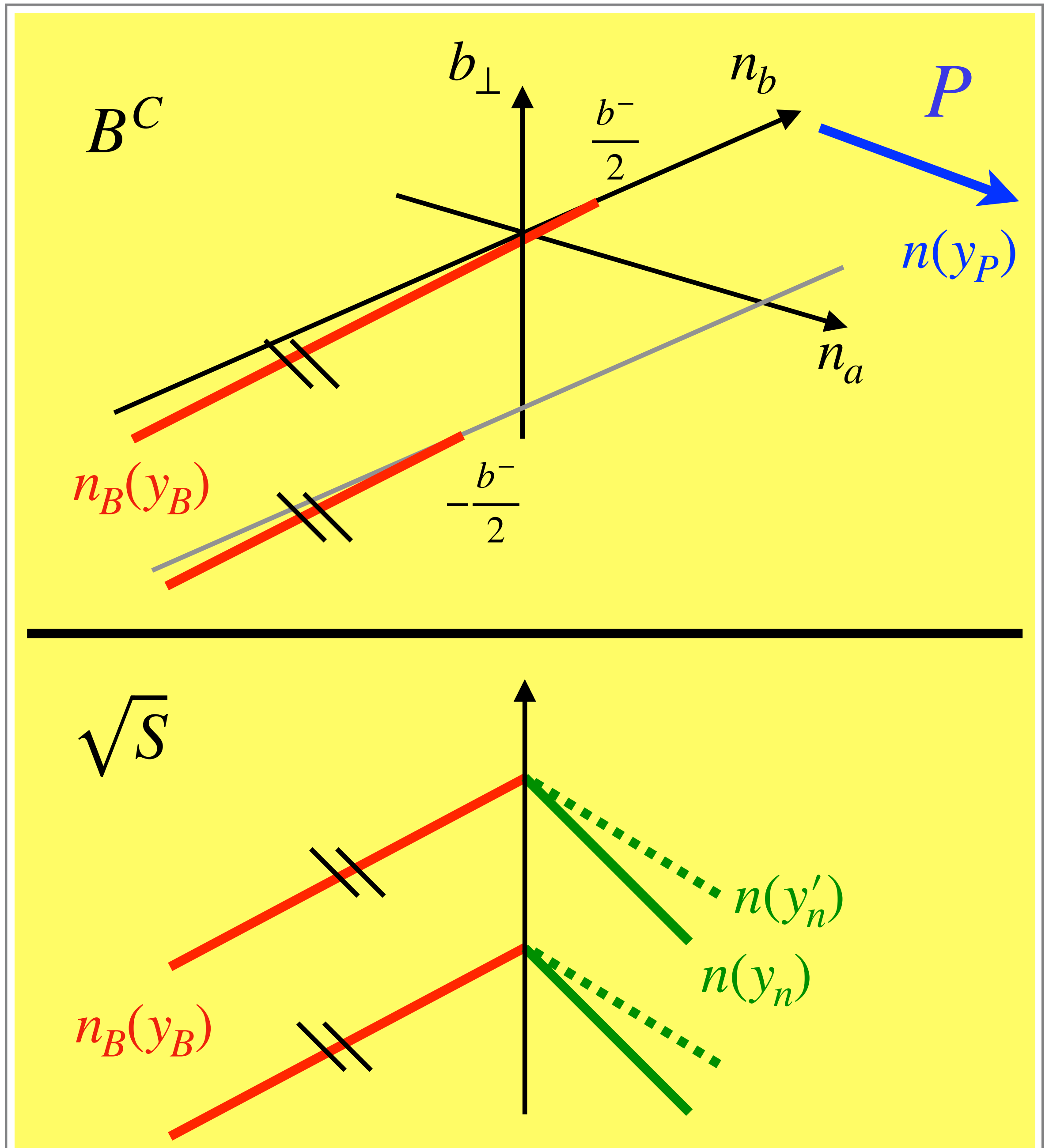
Rapidity divergences $\propto e^{-\gamma^q_\zeta(b_T,\epsilon)(y_P - y_B)},\ e^{-\gamma^q_\zeta(b_T,\epsilon)(2y_n - 2y_B)}$

$$\zeta = 2(xP^+e^{-y_n})^2 = x^2 m_h^2 e^{2(y_P - y_n)}$$

Large Rapidity (LR) scheme

$b = (0, b^-, b_\perp)$ $\delta = b^- n_b$

Ebert, Schindler, Stewart and YZ, 2201.08401.



$$n^\mu(y_n) \equiv (1, -e^{-2y_n}, 0_\perp)$$

Reversed order of limits:

$$f_{i/h}^{\mathrm{LR}}(x, \vec{b}_T, \mu, \zeta, y_P - y_B)$$

$$= \lim_{-y_B \gg 1} \lim_{\epsilon \to 0} \frac{Z_{\mathrm{UV}}^B(\epsilon, \mu)}{\sqrt{Z_{\mathrm{UV}}^S(\epsilon, \mu, 2y_n - 2y_B)}}$$

$$\times \frac{B_{i/h}^C(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}}$$

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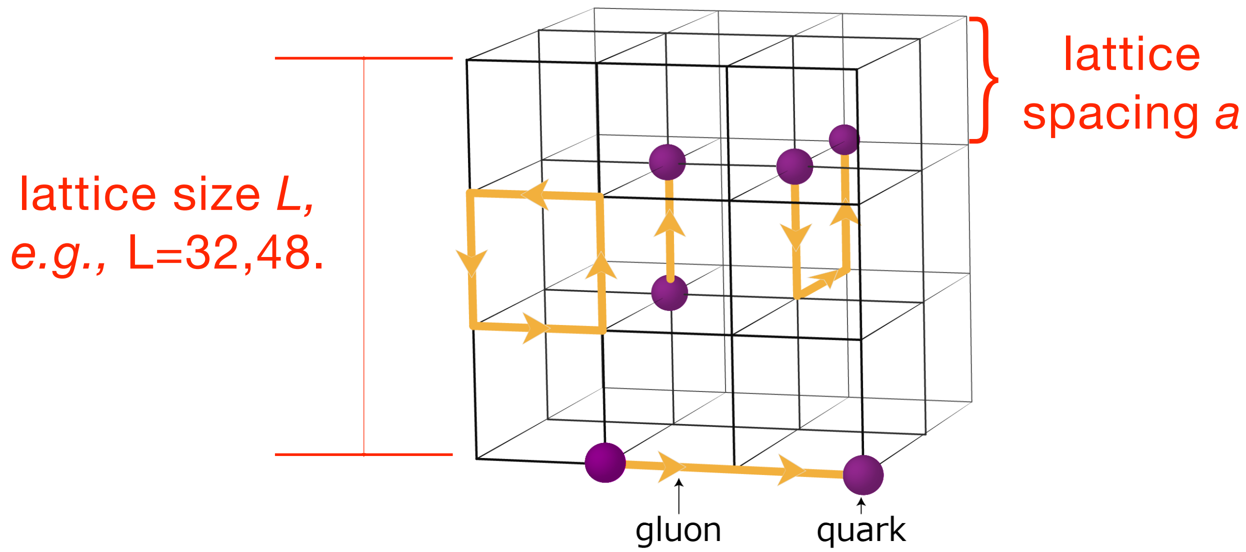
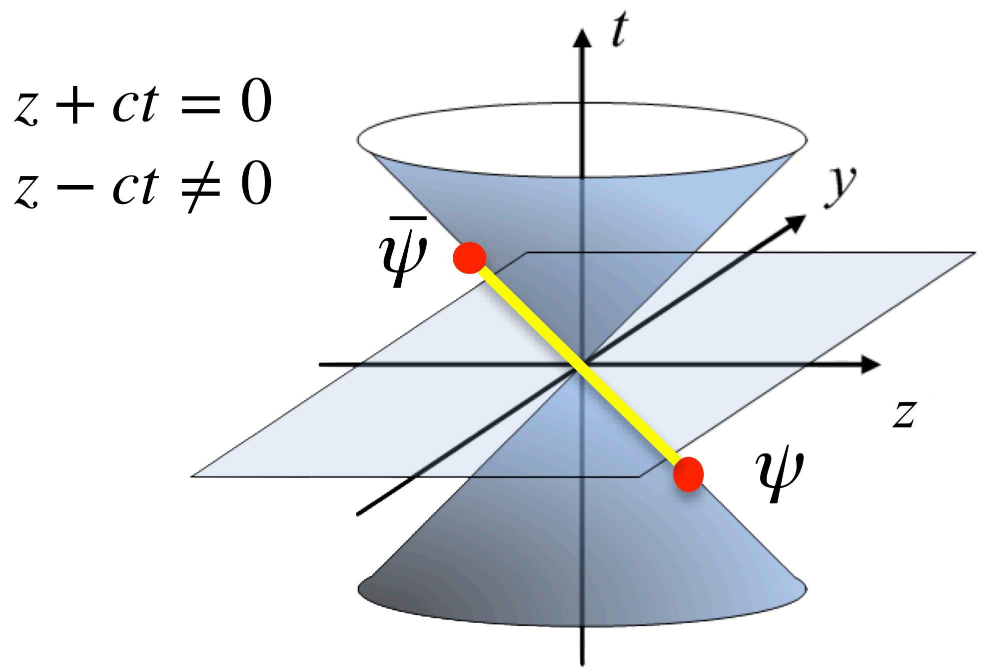
Lattice QCD

Lattice gauge theory: a systematically improvable approach to solve non-perturbative QCD.





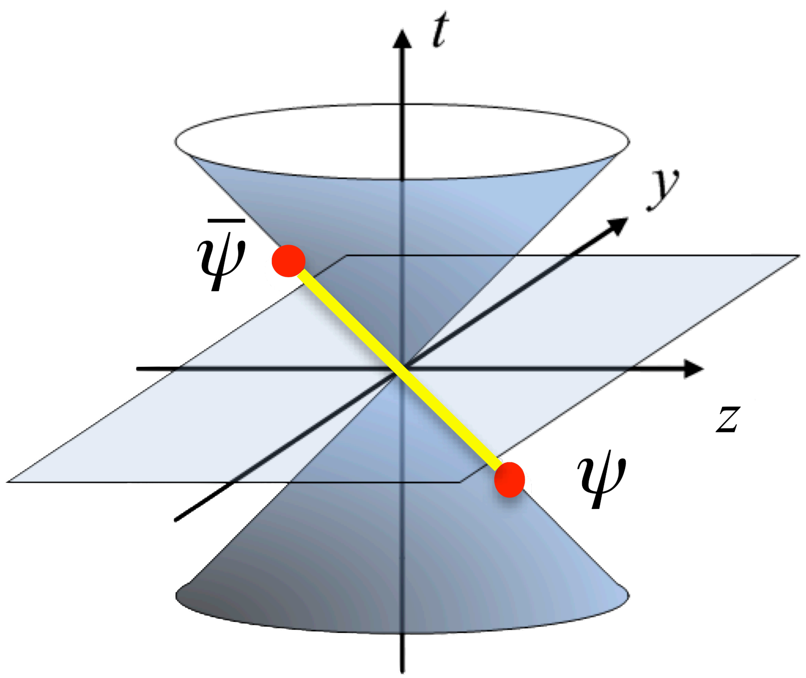
Imaginary time: $t \to i\tau$ $\quad O(i\tau) \stackrel{?}{\to} O(t)$

Simulating real-time dynamics has been extremely difficult due to the issue of analytical continuation. 🙁

Large-Momentum Effective Theory (LaMET)

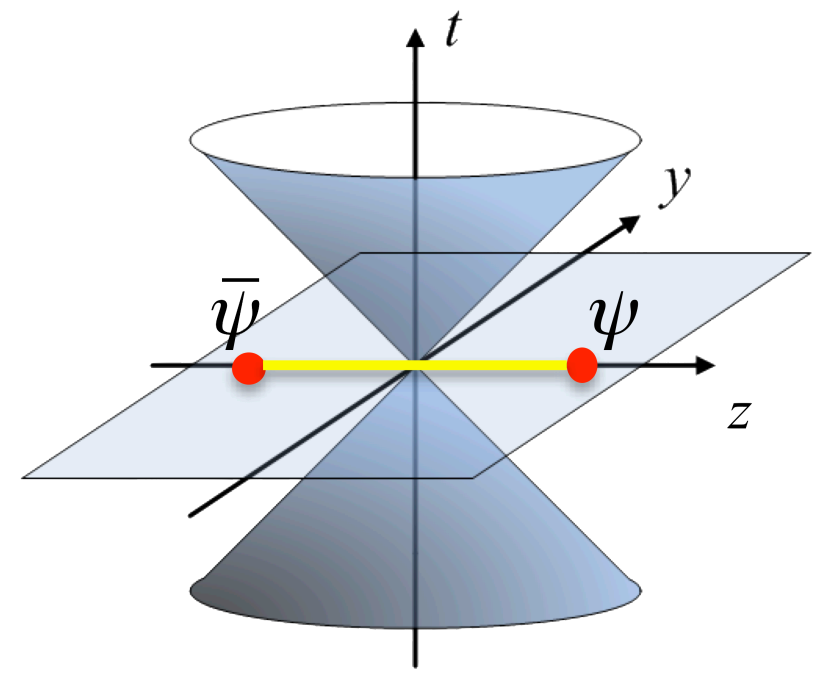
X. Ji, PRL 110 (2013)

$$z + ct = 0, \quad z - ct \neq 0$$



PDF $f(x)$:
Cannot be calculated on the lattice

$$t = 0, \quad z \neq 0$$

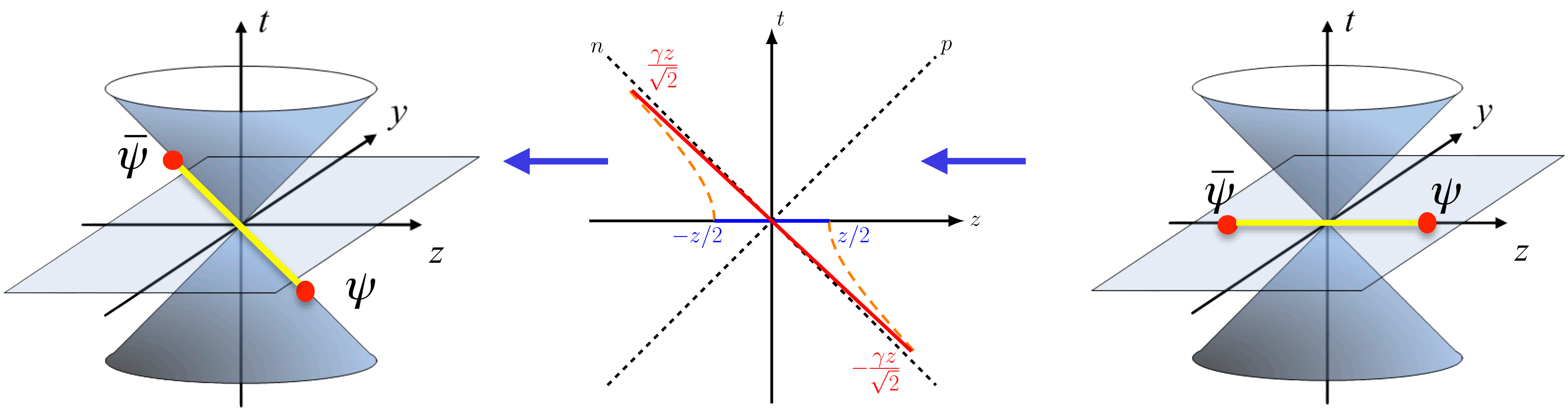


Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

Large-Momentum Effective Theory (LaMET)

X. Ji, PRL 110 (2013)

$z + ct = 0, \quad z - ct \neq 0$ | Related by Lorentz boost | $t = 0, \quad z \neq 0$



PDF $f(x)$:
Cannot be calculated on the lattice

Quasi-PDF $\tilde{f}(x, P^z)$:
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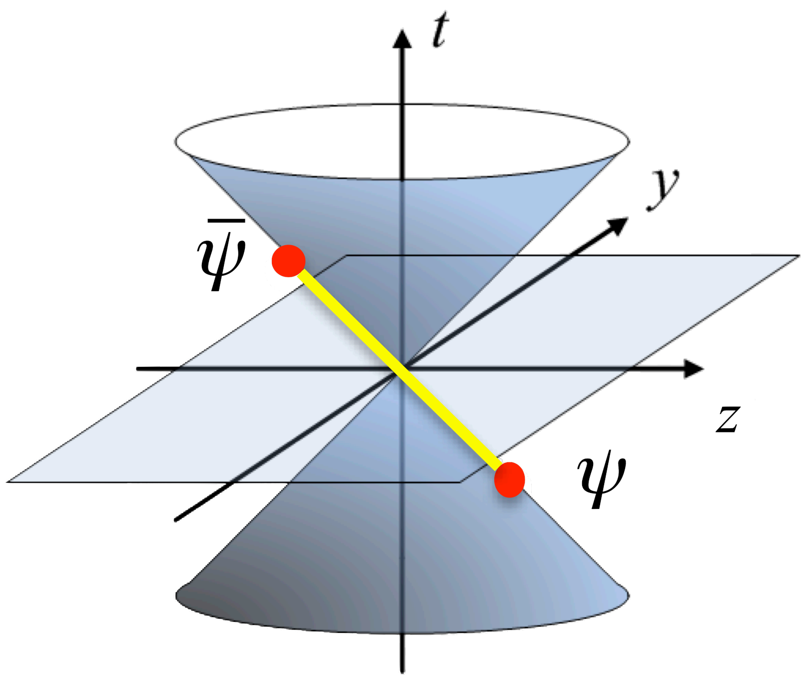
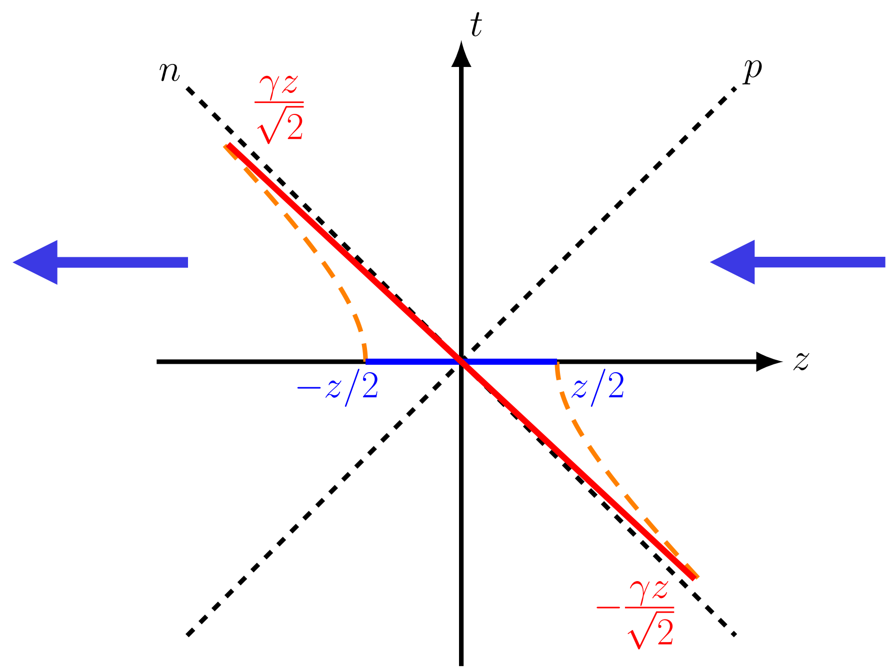
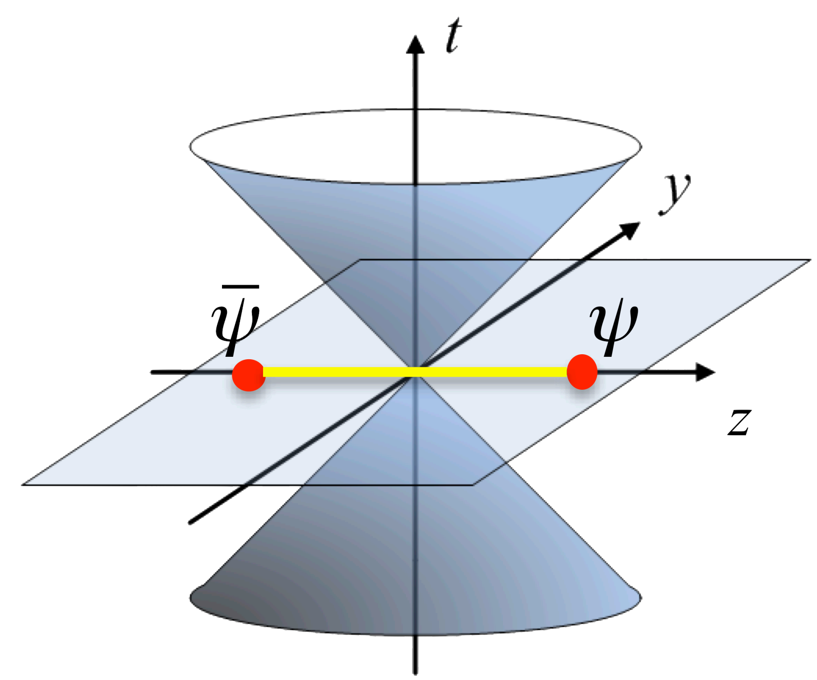


Related by Lorentz boost



$t = 0, \quad z \neq 0$



PDF $f(x)$:
Cannot be calculated on the lattice

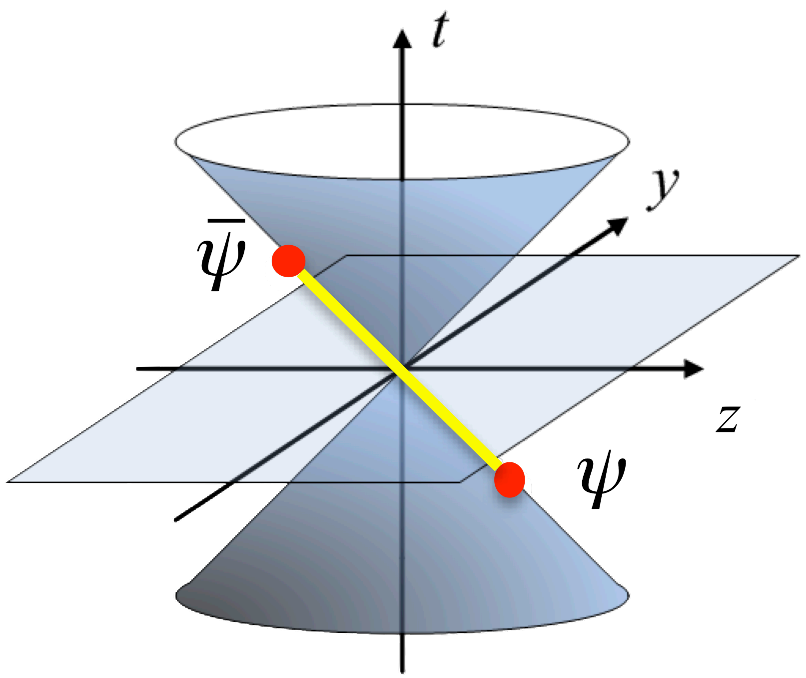
$$\lim_{P^z \to \infty} \tilde{f}(x, P^z) \stackrel{?}{=} f(x)$$

Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

Large-Momentum Effective Theory (LaMET)

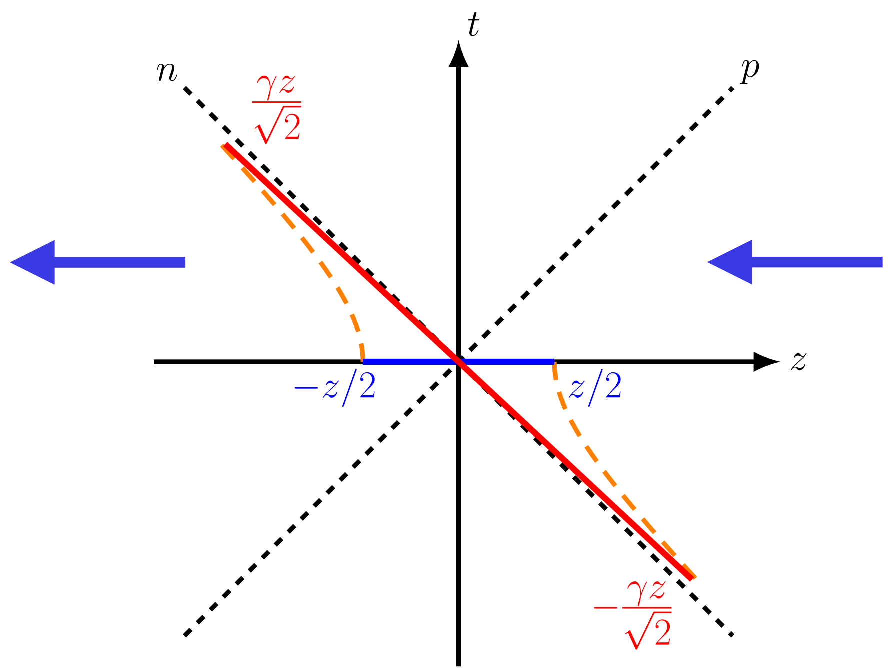
X. Ji, PRL 110 (2013)

$z + ct = 0, \quad z - ct \neq 0$



PDF $f(x)$:
Cannot be calculated on the lattice

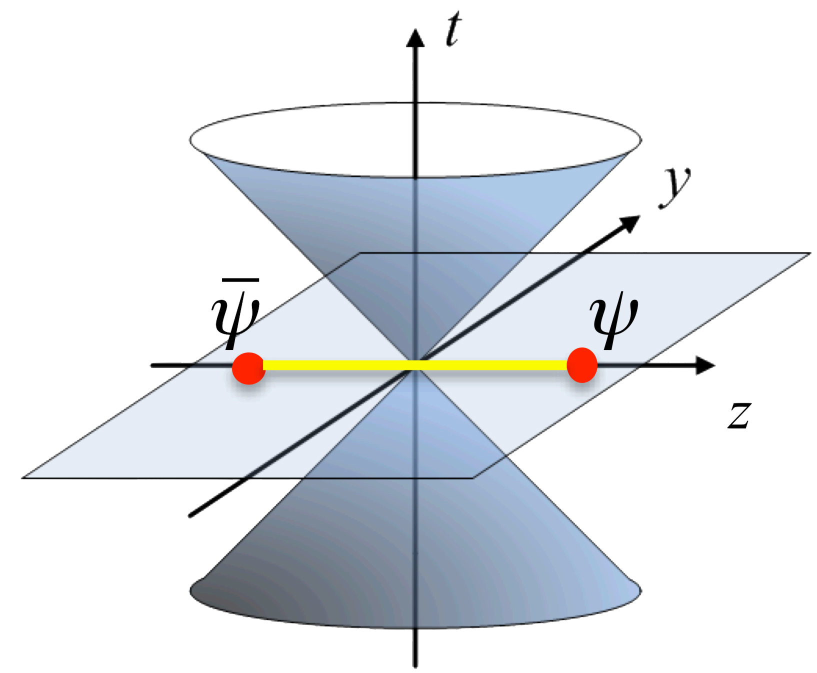
Related by Lorentz boost



$$\lim_{P^z \to \infty} \tilde{f}(x, P^z) \stackrel{?}{=} f(x)$$

✘

$t = 0, \quad z \neq 0$



Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

Large-Momentum Effective Theory (LaMET)

- Quasi-PDF: $P^z \ll \Lambda$;

Λ: the ultraviolet lattice cutoff, $\sim 1/a$

- PDF: $P^z = \infty$, implying $P^z \gg \Lambda$.
  - The limits $P^z \ll \Lambda$ and $P^z \gg \Lambda$ are not usually exchangeable;
  - For $P^z \gg \Lambda_{\mathrm{QCD}}$, the infrared (nonperturbative) physics is not affected, which allows for an effective field theory matching.

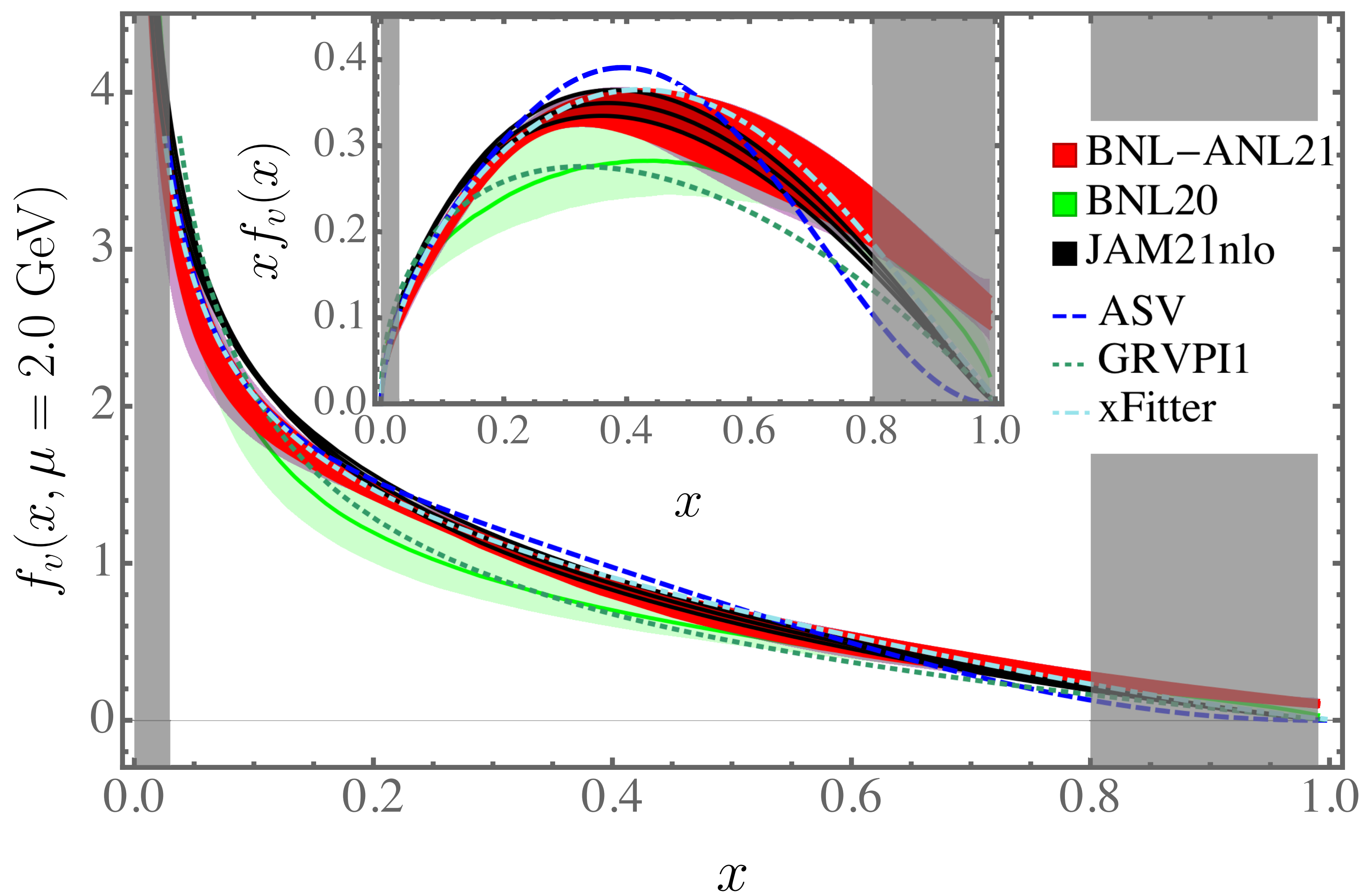
$$\tilde{f}(x, P^z, \Lambda) = \boxed{C\left(x, P^z/\mu, \Lambda/P^z\right)} \otimes f(x, \mu) + O\left(\frac{\Lambda_{\mathrm{QCD}}^2}{P_z^2}\right)$$

Perturbative matching — Power corrections

- X. Ji, PRL 110 (2013); SCPMA57 (2014).
- X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, RMP 93 (2021).

LaMET calculation of the collinear PDFs

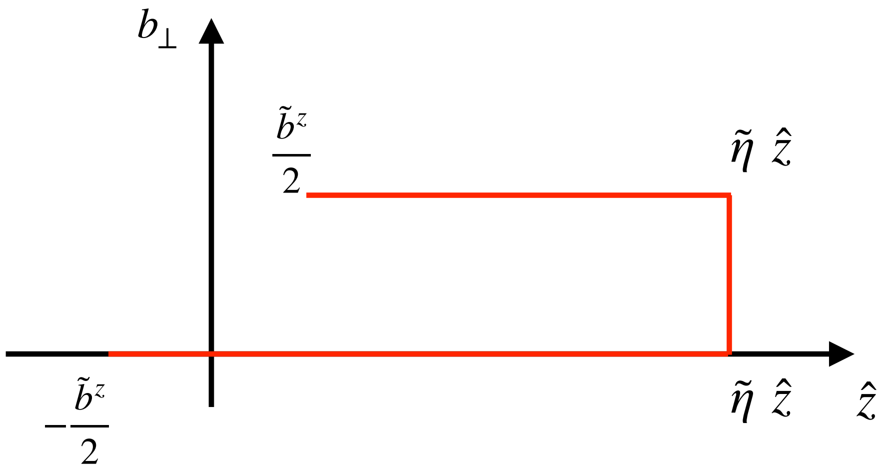
A state-of-the-art calculation of the pion valence quark PDF with fine lattices, large momentum and NNLO matching:



Gao, Hanlon, Mukherjee, Petreczky, Scior, Syritsyn and YZ, 2112.02208.

Quasi-TMD

$\tilde{b}^\mu = (0, b_T^x, b_T^y, \tilde{b}^z)$ $\quad$ $\delta = \tilde{b}^z \hat{z} = (0,0,0,\tilde{b}^z)$

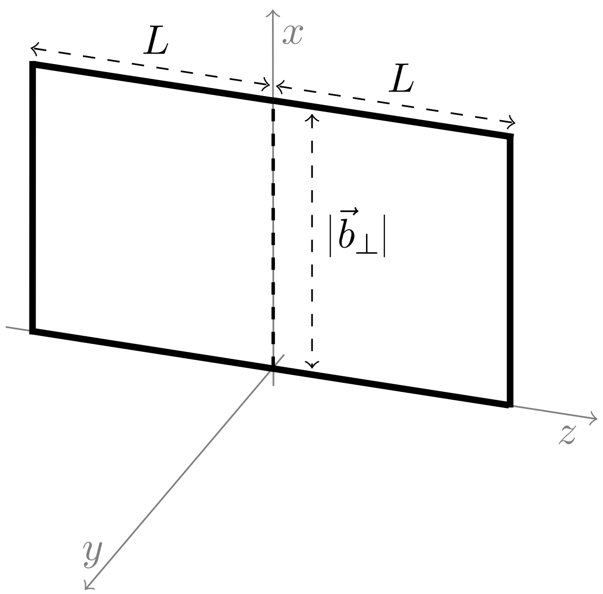


Quasi-beam function:

$$\tilde{B}_{i/h}(x, \overrightarrow{b}_T, a, \tilde{\eta}, x\tilde{P}^z)\,, \quad \tilde{P}^z \gg m_h$$

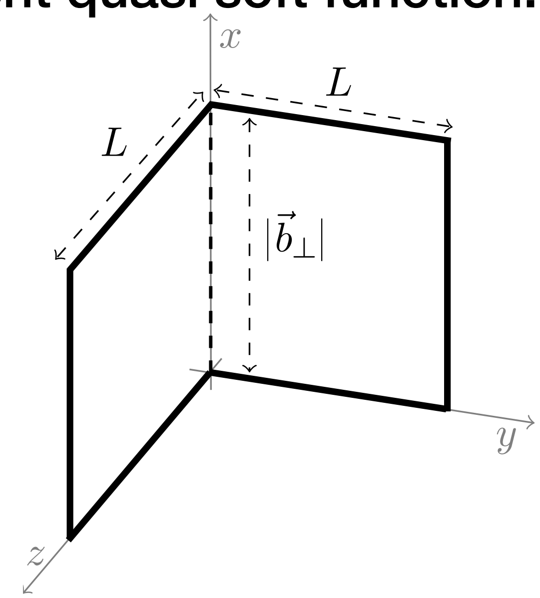
Quasi-soft function?

Naive quasi soft function:



Neither can be boosted to the soft function in TMD factorization. 🙁

Bent quasi soft function:



- Ji, Jin, Yuan, Zhang and YZ, PRD99 (2019);
- M. Ebert, I. Stewart, YZ, PRD99 (2019), JHEP09 (2019).

- Ji, Sun, Xiong and Yuan, PRD91 (2015);
- M. Ebert, I. Stewart, YZ, JHEP09 (2019).

Quasi-TMD

Naive Quasi-TMD:

Linear power divergence

$$\tilde{f}_{i/h}(x, \vec{b}_T, \mu, x\tilde{P}^z) = \lim_{\substack{a \to 0 \\ \tilde{\eta} \to \infty}} Z_{\mathrm{uv}}(a, \mu) \frac{\tilde{B}_{i/h}(x, \vec{b}_T, a, \tilde{\eta}, x\tilde{P}^z)}{\sqrt{\tilde{S}^{\mathrm{naive}}(b_T, a, \tilde{\eta})}} \qquad \propto (2\tilde{\eta} + b_T)/a$$

Conjectured factorization relation to the physical TMD:

Non-perturbative factor

Rapidity evolution/ Collins-Soper kernel

$$\tilde{f}_{\mathrm{ns}}^{\mathrm{naive}}(x, \vec{b}_T, \mu, x\tilde{P}^z) = C_{\mathrm{ns}}(x\tilde{P}^z, \mu)\; g_S^q(b_T, \mu)\; \exp\left[\frac{1}{2}\gamma_\zeta^q(b_T, \mu)\ln\frac{(2x\tilde{P}^z)^2}{\zeta}\right]$$

$$\times f_{\mathrm{ns}}(x, \vec{b}_T, \mu, \zeta)\left\{1 + \mathcal{O}\left[\frac{1}{(x\tilde{P}^z b_T)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(x\tilde{P}^z)^2}\right]\right\}$$

Matching coefficient

M. Ebert, I. Stewart, YZ, PRD99 (2019), JHEP09 (2019).

Quasi-TMD

Proposal to calculate the soft sector from lattice:

$$\frac{\tilde{f}^{\text{naive}}_{\text{ns}}(x, \overrightarrow{b}_T, \mu, \tilde{P}^z)}{\sqrt{S^q_r(b_T, \mu)}} = C_{\text{ns}}\left(\mu, x\tilde{P}^z\right) \exp\left[\frac{1}{2}\gamma^q_\zeta(\mu, b_T)\ln\frac{(2x\tilde{P}^z)^2}{\zeta}\right]$$

$$\times f_{\text{ns}}(x, \overrightarrow{b}_T, \mu, \zeta)\left\{1 + \mathcal{O}\left[\frac{1}{(x\tilde{P}^z b_T)^2}, \frac{\Lambda^2_{\text{QCD}}}{(x\tilde{P}^z)^2}\right]\right\}$$

Reduced soft factor ✓

Ji, Liu and Liu, NPB 955 (2020), PLB 811 (2020).

Proof of factorization relation:

- Leading-region of Feynman diagrams

  Ji, Liu and Liu, NPB 955 (2020), PLB 811 (2020), and the complete analysis in preparation.

- Analysis of lattice-friendly TMD correlators

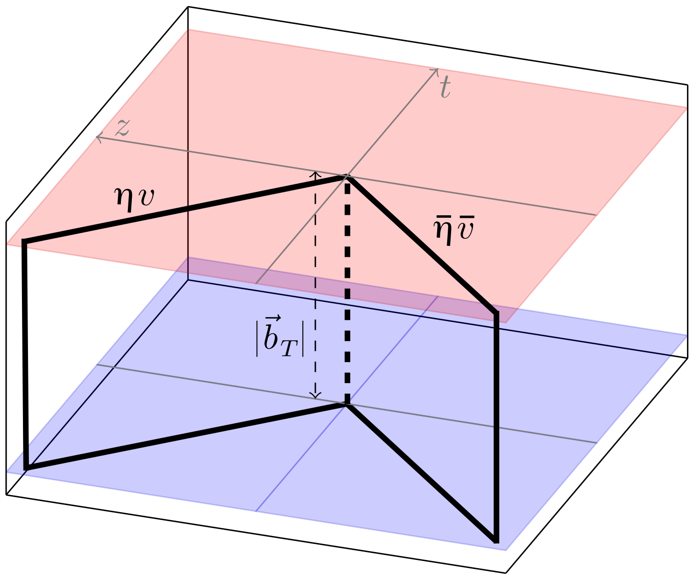
  A. Vladimirov and A. Schäfer, PRD 101 (2020).

- Proof using effective field theory (LaMET).

  Ebert, Schindler, Stewart and YZ, 2201.08401.

Quasi-TMD

Desired quasi soft factor:



$$\tilde{S}(b_T, a, \tilde{\eta}, y_A, y_B) = S^C\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{|n_A(y_A)|}, -\tilde{\eta}\frac{n_B(y_B)}{|n_B(y_B)|}\right]$$

Ebert, Schindler, Stewart and YZ, 2201.08401.

Quasi-TMD:

$$\tilde{f}_{i/h}(x, \overrightarrow{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z) = \lim_{a\to 0} Z_{\mathrm{uv}}(a, \mu)\frac{\tilde{B}_{i/h}(x, \overrightarrow{b}_T, a, \tilde{\eta}, x\tilde{P}^z)}{\sqrt{\tilde{S}^R(b_T, a, \tilde{\eta}, y_A, y_B)}}$$

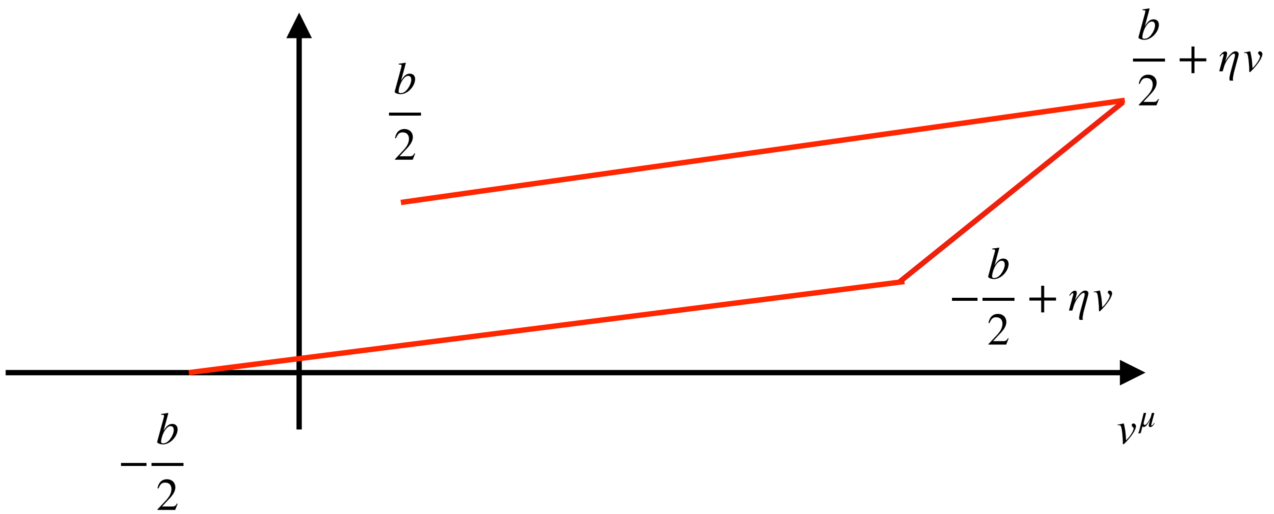
Lorentz-invariant approach: MHENS scheme

Musch-Engelhardt-Negele-Hägler-Schäfer (MHENS) scheme:

$b = (0, b^-, b_\perp)$ $\delta = 0$

Can always use Lorentz invariance of $P \cdot b$ to boost to a frame where

$$\tilde{b}^\mu = (0, b_T^x, b_T^y, \tilde{b}^z)$$



$$B_{q/h}^{\mathrm{MHENS}\ [\Gamma]}(x, \vec{b}_T, P, a, \eta, v)$$

Hägler, Musch, Engelhardt, Negele, Schäfer, et al.,
EPL88 (2009), PRD83 (2011), PRD85 (2012), PRD93 (2016), 1601.05717, PRD96 (2017).

Ratios of TMDs can be calculated without the soft function.

Summary of the continuum and lattice schemes

	TMD	Beam function	Soft function
Collins	$\lim_{\epsilon\to 0} Z_{\rm UV}^R \lim_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega_{q/h}^{[\gamma^+]}\left[b,P,\epsilon,-\infty n_B(y_B),b^- n_b\right]$	$S^R\left[b_\perp,\epsilon,-\infty n_A(y_A),-\infty n_B(y_B)\right]$
LR	$\lim_{-y_B\gg 1}\lim_{\epsilon\to 0} Z_{\rm UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega_{q/h}^{[\gamma^+]}\left[b,P,\epsilon,-\infty n_B(y_B),b^- n_b\right]$	$S^R\left[b_\perp,\epsilon,-\infty n_A(y_A),-\infty n_B(y_B)\right]$
Quasi	$\lim_{a\to 0} Z_{\rm UV} \frac{B_{i/h}}{\sqrt{\tilde S^R}}$	$\Omega_{q/h}^{[\gamma^{0,z}]}(\tilde b,\tilde P,a,\tilde\eta\hat z,\tilde b^z\hat z)$	$S^R\left[b_\perp,a,-\tilde\eta\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert},-\tilde\eta\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
MHENS		$\Omega_{q/h}^{[\Gamma]}(b,P,a,\eta v,0)$	

	Collins / LR	Quasi	MHENS
b^μ	$(0,b^-,b_\perp)$	$(0,b_T^x,b_T^y,\tilde b^z)$	$(0,b_T^x,b_T^y,\tilde b^z)$
v^μ	$(-e^{2y_B},1,0_\perp)$	$(0,0,0,-1)$	$(0,v^x,v^y,v^z)$
δ^μ	$(0,b^-,0_\perp)$	$(0,0,0,\tilde b^z)$	$(0,0,0_\perp)$
P^μ	$\frac{m_h}{\sqrt{2}}(e^{y_P},e^{-y_P},0_\perp)$	$m_h(\cosh y_{\tilde P},0,0,\sinh y_{\tilde P})$	$m_h\left(\cosh y_P,\frac{P^x}{m_h},\frac{P^y}{m_h},\sinh y_P\right)$

Outline

- Introduction to TMDs
- Lattice TMDs
  - LaMET and Quasi-TMDs
  - Lorentz-invariant approach (MHENS scheme)
- Relation between lattice and continuum TMDs
- First lattice results
  - Collins-Soper kernel for TMD evolution
  - Soft function

Lorentz-invariant TMD variables

Collins/LR/MHENS beam function

$$B_{q/h}(x, \overrightarrow{b}_T, \epsilon, \eta, \ldots) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \Omega_{q/h}^{[\gamma^+]} = \int \frac{d(P\cdot b)}{2\pi} e^{-ix(P\cdot b)} \Phi_{q/h}(P\cdot b, b^2, \eta^2 v^2, \ldots)$$

Quasi-/MHENS beam function

$$\tilde{B}_{q/h}^{\tilde{\Gamma}}(x, \overrightarrow{b}_T, \epsilon, \tilde{\eta}, \ldots) = N_{\tilde{\Gamma}} \int \frac{d\tilde{b}^z}{2\pi} e^{i\tilde{b}^z(x\tilde{P}^z)} \Omega_{q/h}^{[\tilde{\Gamma}]} = \int \frac{d(\tilde{P}\cdot\tilde{b})}{2\pi} e^{-ix(\tilde{P}\cdot\tilde{b})} \Phi_{q/h}(\tilde{P}\cdot\tilde{b}, \tilde{b}^2, \tilde{\eta}^2\hat{z}^2, \ldots)$$

- 10 Lorentz invariant scalars;
- Reduces to 6 when $\delta = 0$ in the MHENS scheme.

Lorentz-invariant TMD variables

$\sinh(y_P - y_B) = \sinh y_{\tilde{P}}$

$\Rightarrow y_{\tilde{P}} = y_P - y_B$

$y_B \to -\infty \Rightarrow y_{\tilde{P}} \to -\infty$

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2 \vec{v}^2$
$P \cdot b$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \mathrm{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \mathrm{sgn}(\eta)$	$\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh(y_P - y_B)\, \mathrm{sgn}(\eta)$	$\sinh y_{\tilde{P}}\, \mathrm{sgn}(\eta)$	$\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$
$\frac{\delta^2}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1	0
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

Lorentz-invariant TMD variables

$y_P\,,\, b^-$ finite

$y_{\tilde{P}} \to -\infty \Rightarrow \tilde{b}^z \to 0$

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2 \vec{v}^{\,2}$
$P\cdot b$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
$\frac{b\cdot(\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T}\,\mathrm{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}}\,\mathrm{sgn}(\eta)$	$\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$
$\frac{P\cdot(\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$	$\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$	$\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$
$\frac{\delta^2}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b\cdot\delta}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P\cdot\delta}{P\cdot b}$	1	1	0
$\frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

Lorentz-invariant TMD variables

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2-(\tilde{b}^z)^2$	$-b_T^2-(\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2\vec{v}^{\,2}$
$P\cdot b$	$\frac{m_h}{\sqrt{2}}b^- e^{y_P}$	$-m_h\tilde{b}^z\sinh y_{\tilde{P}}$	$m_h\sinh y_P\tilde{b}^z+P^x b_T^x+P^y b_T^y$
$\frac{b\cdot(\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\,b_T}\,\mathrm{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2+b_T^2}}\,\mathrm{sgn}(\eta)$	$\frac{b_T^x v^x+b_T^y v^y+\tilde{b}^z v^z}{\sqrt{v_T^2+(v^z)^2}\sqrt{b_T^2+(\tilde{b}^z)^2}}$
$\frac{P\cdot(\eta v)}{\sqrt{P^2\lvert\eta v\rvert^2}}$	$\sinh(y_P-y_B)\,\mathrm{sgn}(\eta)$	$\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$	$\frac{P^x v^x+P^y v^y+m_h v^z\sinh y_P}{\sqrt{v_T^2+(v^z)^2}\sqrt{m_h^2+P_x^2+P_y^2}}$
$\frac{\delta^2}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2+(\tilde{b}^z)^2}$	0
$\frac{b\cdot\delta}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2+(\tilde{b}^z)^2}$	0
$\frac{P\cdot\delta}{P\cdot b}$	1	1	0
$\frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

Relating LR and quasi- TMDs

- Collins/LR and quasi-beam functions are in the same class of correlators if

$$y_{\tilde{P}} = y_P - y_B$$

- So we can define the quasi-TMD with the Collins soft function

$$\tilde{f}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta})$$
$$= \int \frac{\mathrm{d}(\tilde{P}\cdot\tilde{b})}{2\pi} e^{-\mathrm{i}x(\tilde{P}\cdot\tilde{b})} \lim_{\epsilon\to 0} Z_{\mathrm{uv}}^q(\mu, \epsilon, y_n - y_B) \frac{\Omega_{q_i/h}(\tilde{b}, \tilde{P}, \epsilon, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})}{\sqrt{\tilde{S}^q(b_T, \epsilon, \tilde{\eta}, 2y_n, 2y_B)}}$$

$$\tilde{\zeta} = x^2 m_h^2 e^{2\tilde{y}_P + 2y_B - 2y_n} \qquad \zeta \equiv x^2 m_h^2 e^{2y_P - 2y_n}$$

Therefore,

$$\lim_{y_B \ll -1} \tilde{f}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\eta}, \tilde{\zeta}, x\tilde{P}^z) = \lim_{y_B \ll -1} f_{q_i/h}^{\mathrm{LR}}\Big(x, \vec{b}_T, \mu, -\frac{\tilde{\eta}}{2} e^{-y_B}, \tilde{\zeta}, y_P - y_B\Big)$$

Relating LR and Collins TMDs

Collins scheme:

$$f^{C}_{i/h}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\rm UV} \lim_{y_B \to -\infty} \frac{B_{i/h}}{\sqrt{S_C}}$$

LR scheme:

$$f^{\rm LR}_{i/h}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) = \lim_{-y_B \gg 1} \lim_{\epsilon\to 0} Z'_{\rm UV} \frac{B_{i/h}}{\sqrt{S_C}}$$

- Large $-y_B$ corresponds to a hard momentum scale $\zeta_{\rm LR} = 4x^2M^2 \sinh(y_P - y_B)$.
- Exchange of $\epsilon \to 0$ and $\zeta_{\rm LR} \to \infty$ should not affect the infrared physics, so the difference between the orders of limits is compensated by perturbative matching !

$$f^{C}_{i/h}(x, \vec{b}_T, \mu, \zeta) = C^{-1}\left(\frac{\zeta_{\rm LR}}{\mu^2}\right) f^{\rm LR}_{i/h}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) + \mathcal{O}(y_B^{-k} e^{y_B})$$

Verified at 1-loop ✔

- **"LaMET"**, Ji, PRL 110 (2013); SCPMA57 (2014); Ji, Liu, Liu, Zhang and YZ, RMP 93 (2021);
- Collins, 2011 book, Ch. 10, on Sudakov form factors.

Relating LR and Collins TMDs

...linear and soft factors. ... associated with the Wilson lines in light-like directions cancel between the numerators and denominators. In effect,

$$A^{\text{basic}} = \underset{y_{u_2}\to-\infty}{\text{“lim”}} \frac{A^{\text{unsub}}(y_{p_A} - y_{u_2})}{S(y_1 - y_{u_2})}, \tag{10.95}$$

and similarly for B^{basic}. However, there is a non-uniformity in taking the infinite rapidity ... which impacts calculations. As indicated by the ...

Co...

- La...).
- Ex... ne dif... ng !

If we reversed the limits, we would need to compensate by an extra hard factor, e.g.,

$$A^{\text{basic}} = \lim_{y_{u_2}\to-\infty}\left[\lim_{n\to4} \frac{A^{\text{unsub}}(y_{p_A} - y_{u_2})}{S(y_1 - y_{u_2})} \tilde{Z}_A(\zeta_{A,n_1}/\mu^2, y_1 - y_{u_2}, g(\mu), \epsilon)\right]. \tag{10.97}$$

The factor $\tilde{Z}_A$ is to be adjusted so that we get the same results as in (10.94a). Now the non-uniformity of the limits $n \to 4$ and of infinite Wilson-line rapidities only concerns the limit of infinitely large transverse momentum; for $n < 4$, the limits can be exchanged. Thus the factor $\tilde{Z}_A$ is a pure UV factor, and can be regarded as a kind of generalized UV renormalization factor, chosen to make a renormalization prescription that agrees with the combination of $\overline{\text{MS}}$ renormalization and the opposite order of the limits.

Within the context of low-order ...

, Liu,

Relating LR and Collins TMDs

Collins scheme:

$$f^{C}_{i/h}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\rm UV} \lim_{y_B \to -\infty} \frac{B_{i/h}}{\sqrt{S_C}}$$

LR scheme:

$$f^{\rm LR}_{i/h}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) = \lim_{-y_B \gg 1} \lim_{\epsilon\to 0} Z'_{\rm UV} \frac{B_{i/h}}{\sqrt{S_C}}$$

- Large $-y_B$ corresponds to a hard momentum scale $\zeta_{\rm LR} = 4x^2M^2 \sinh(y_P - y_B)$.
- Exchange of $\epsilon \to 0$ and $\zeta_{\rm LR} \to \infty$ should not affect the infrared physics, so the difference between the orders of limits is compensated by perturbative matching !

$$f^{C}_{i/h}(x, \vec{b}_T, \mu, \zeta) = C^{-1}\left(\frac{\zeta_{\rm LR}}{\mu^2}\right) f^{\rm LR}_{i/h}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) + \mathcal{O}(y_B^{-k} e^{y_B})$$

Verified at 1-loop ✔

- **"LaMET"**, Ji, PRL 110 (2013); SCPMA57 (2014); Ji, Liu, Liu, Zhang and YZ, RMP 93 (2021);
- Collins, 2011 book, Ch. 10, on Sudakov form factors.

Matching between quasi- and Collins TMDs

$$\lim_{\tilde{\eta}\to\infty} \tilde{f}_{q/h}(x, \vec{b}_T, \mu, \zeta, x\tilde{P}^z, \tilde{\eta}) = C\left(\frac{\tilde{\zeta}_z}{\mu^2}\right) f^C_{i/h}(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}(y_{\tilde{P}}^{-k} e^{-y_{\tilde{P}}})$$

$$\tilde{\zeta}_z = (2x\tilde{P}^z)^2 = \zeta_{\rm LR}$$

Moreover,

$$\tilde{f}_{q/h} = \frac{B_{q/h}}{\sqrt{S_C\big(b_T, \mu, \tilde{\eta}, 2y_n, 2y_B\big)}} \longrightarrow$$

Not directly calculable on the lattice

Naive quasi soft function, lattice calculable ⟵

$$= \frac{B_{q/h}}{\sqrt{S_C\big(b_T, \mu, \tilde{\eta}, 0, 0\big)}} \; \frac{\sqrt{S_C\big(b_T, \mu, \tilde{\eta}, 0, 0\big)}}{\sqrt{S_C\big(b_T, \mu, \tilde{\eta}, 2y_n, 2y_B\big)}}$$

$$\xrightarrow[y_B \to -\infty]{\tilde{\eta}\to\infty} \frac{e^{-\frac{1}{2}\gamma^q_\zeta(b_T,\mu)\ln\frac{\tilde{\zeta}_z}{\zeta}}}{g^q_S(b_T,\mu)}$$

- **Reduced soft function:** $S_r(b_T, \mu) = [g^q_S(b_T, \mu)]^2$
- **Methods for calculation has been proposed and explored on the lattice.**
    - Ji, Liu and Liu, NPB 955 (2020);
    - Q.-A. Zhang, et al. (LP Collaboration), PRL 125 (2020);
    - Y. Li et al., PRL 128 (2022).

Factorization formulas

①

$$\lim_{\tilde{\eta}\to\infty} \tilde{f}_{q/h}(x, \overrightarrow{b}_T, \mu, \tilde{\eta}, \zeta, x\tilde{P}^z) = C\left(\frac{\tilde{\zeta}_z}{\mu^2}\right) f^C_{i/h}(x, \overrightarrow{b}_T, \mu, \zeta) + \mathcal{O}(y_{\tilde{P}}^{-k} e^{-y_{\tilde{P}}})$$

②

$$\frac{\tilde{f}^{\text{naive}}_{i/h}}{g_S^q(b_T, \mu)} = C\left(\frac{\tilde{\zeta}_z}{\mu^2}\right) e^{\frac{1}{2}\gamma_\zeta^q(b_T,\mu)\ln\frac{\tilde{\zeta}_z}{\zeta}} f^C_{q/h}(x, \overrightarrow{b}_T, \mu, \zeta) + \mathcal{O}(y_{\tilde{P}}^{-k} e^{-y_{\tilde{P}}})$$

↓

$$\mathcal{O}\left(\frac{b_T}{\tilde{\eta}}, \frac{1}{(xb_T\tilde{P}^z)^2}, \frac{1}{\tilde{P}^z\tilde{\eta}}, \frac{\Lambda^2_{\text{QCD}}}{(x\tilde{P}^z)^2}\right)$$

- M. Ebert, I. Stewart and YZ, PRD99 (2019), JHEP09 (2019);
- Ji, Liu and Liu, NPB 955 (2020), PLB 811 (2020).
- A. Vladimirov and A. Schäfer, PRD 101 (2020);
- Ebert, Schindler, Stewart and YZ, 2201.08401.

Implications

- Same proof applies for the gluon quasi TMD and for all spin-dependent quasi TMDs;

Both gluon and singlet quark TMDs are calculable on the lattice!

- No mixing between quarks of different flavors, quark and gluon channels, or different spin structures;

Verified at 1-loop ✔

Ebert, Schindler, Stewart and YZ, JHEP 09 (2020).

- The P^z-evolution;

Calculation of gluon TMDs easier than anticipated!

$$\frac{d}{d\ln(2x\tilde{P}^z)} \ln \lim_{\tilde{\eta}\gg b_T} \tilde{B}^{[\tilde{\Gamma}]}_{q/h}(x, \vec{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z) = \gamma^q_\zeta(b_T, \mu) + \gamma^q_C(2x\tilde{P}^z, \mu)$$

Perturbative

Lattice calculation of the Collins-Soper kernel:

- Ji, Sun, Xiong and Yuan, PRD91 (2015);
- M. Ebert, I. Stewart, YZ, PRD99 (2019).

NLL resummation in the Wilson coefficient: $\gamma^q_C(2x\tilde{P}^z, \mu) = \frac{d}{d\ln(2x\tilde{P}^z)} \ln C_q(x\tilde{P}^z, \mu)$

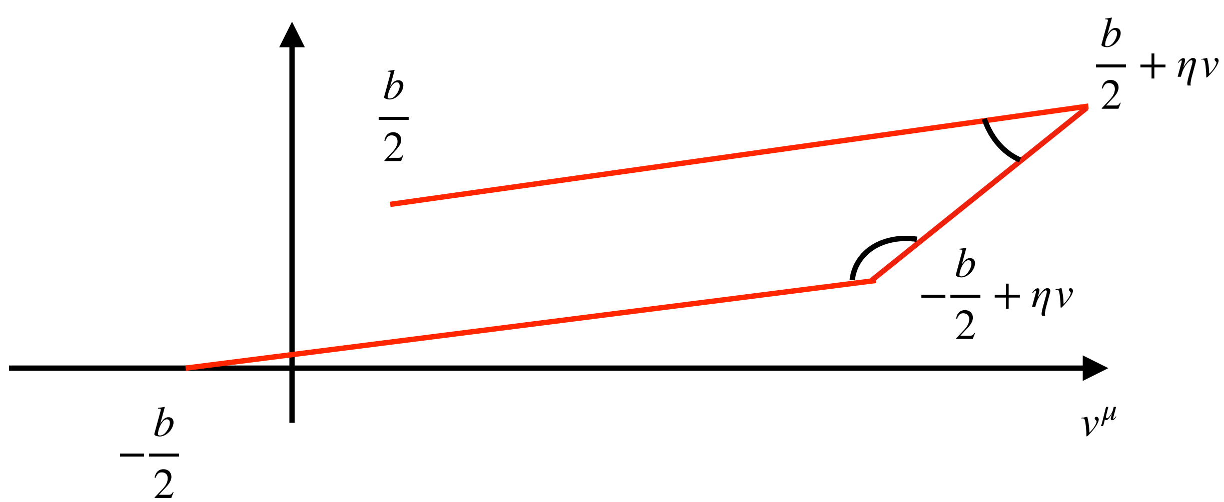
- Ebert, Schindler, Stewart and YZ, 2201.08401.

Implications

- Ratios of TMDs and their x-moments should be calculated in the x-space;

$$\lim_{\tilde{\eta}\to\infty} \frac{\tilde{B}^{[\tilde{\Gamma}_1]}_{q_i/h}(x,\vec{b}_T,\mu,\tilde{\eta},x\tilde{P}^z)}{\tilde{B}^{[\tilde{\Gamma}_2]}_{q_j/h'}(x,\vec{b}_T,\mu,\tilde{\eta},x\tilde{P}^z)} = \frac{f^{[\Gamma_1]}_{q_i/h}(x,\vec{b}_T,\mu,\zeta)}{f^{[\Gamma_2]}_{q_j/h'}(x,\vec{b}_T,\mu,\zeta)}$$

- Factorization for the MHENS TMD.



Additional challenges beyond tree-level renormalization/matching:

- b^z-dependent renormalization

Linear: $\propto \left(2|\eta v| + \sqrt{\tilde{b}_z^2 + b_T^2}\right)/a$

Cusp: $\propto \left[3 - \frac{2\tilde{b}^z}{b_T}\tan^{-1}\frac{b_T}{\tilde{b}^z}\right] \ln(a)$

- b^z-dependent soft function?

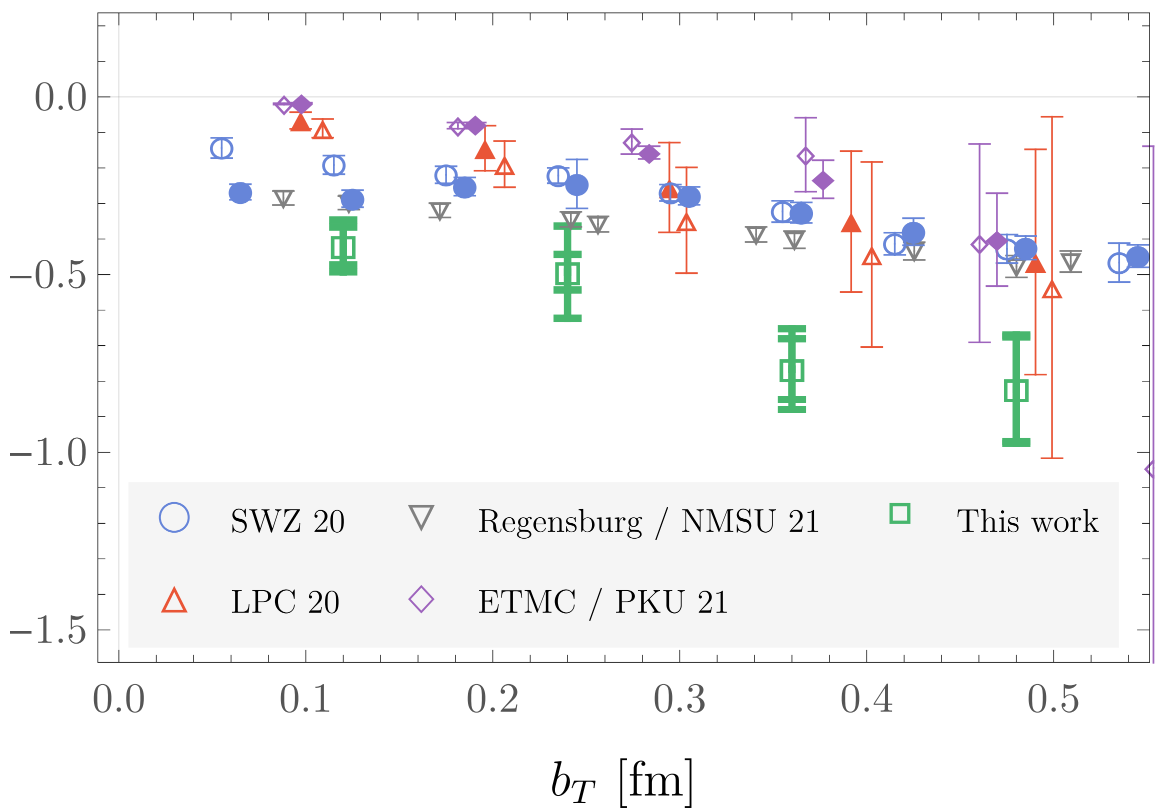
- Ratios of the x-moments of TMDs can be calculated with MHENS beam functions at $\tilde{b}^z = 0$ with tree-level matching;
- With proper lattice renormalization and soft function subtraction, the MHENS scheme should be equivalent to the LR scheme.

Outline

- Introduction to TMDs
- Lattice TMDs
  - LaMET and Quasi-TMDs
  - Lorentz-invariant approach (MHENS scheme)
- Relation between lattice and continuum TMDs
- First lattice results
  - Collins-Soper kernel for TMD evolution
  - Soft function

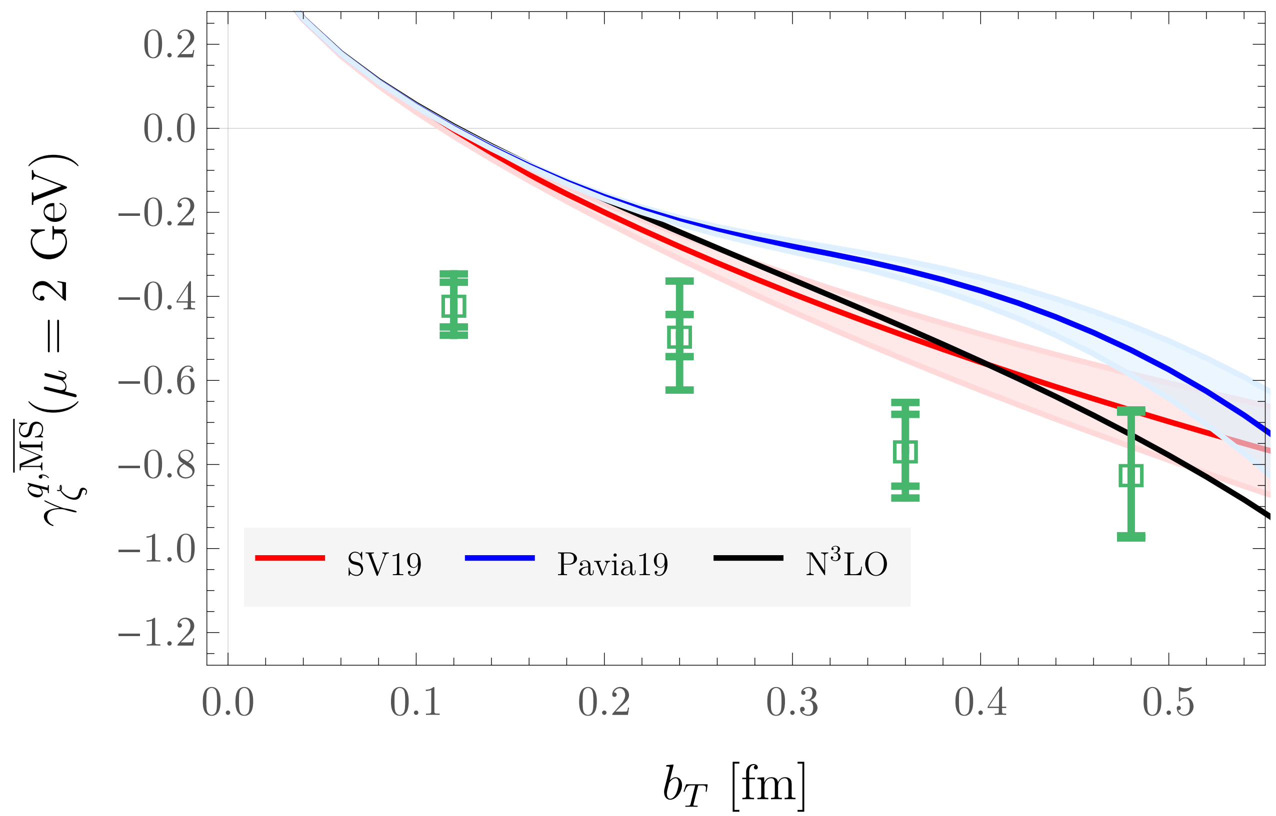
Collins-Soper kernel

Results by different groups with different systematics.



- **SWZ20**, P. Shanahan, M. Wagman and YZ, PRD 102 (2020) ;
- **Regensburg/NMSU21**, Schlemmer, Vladimirov, Zimmerman, Engelhardt, Schäfer, JHEP 08 (2021);
- **LPC20**, Q.-A. Zhang, et al. (LPC), PRL 125 (2020);
- **ETMC/PKU21**, Y. Li et al., PRL 128 (2022).

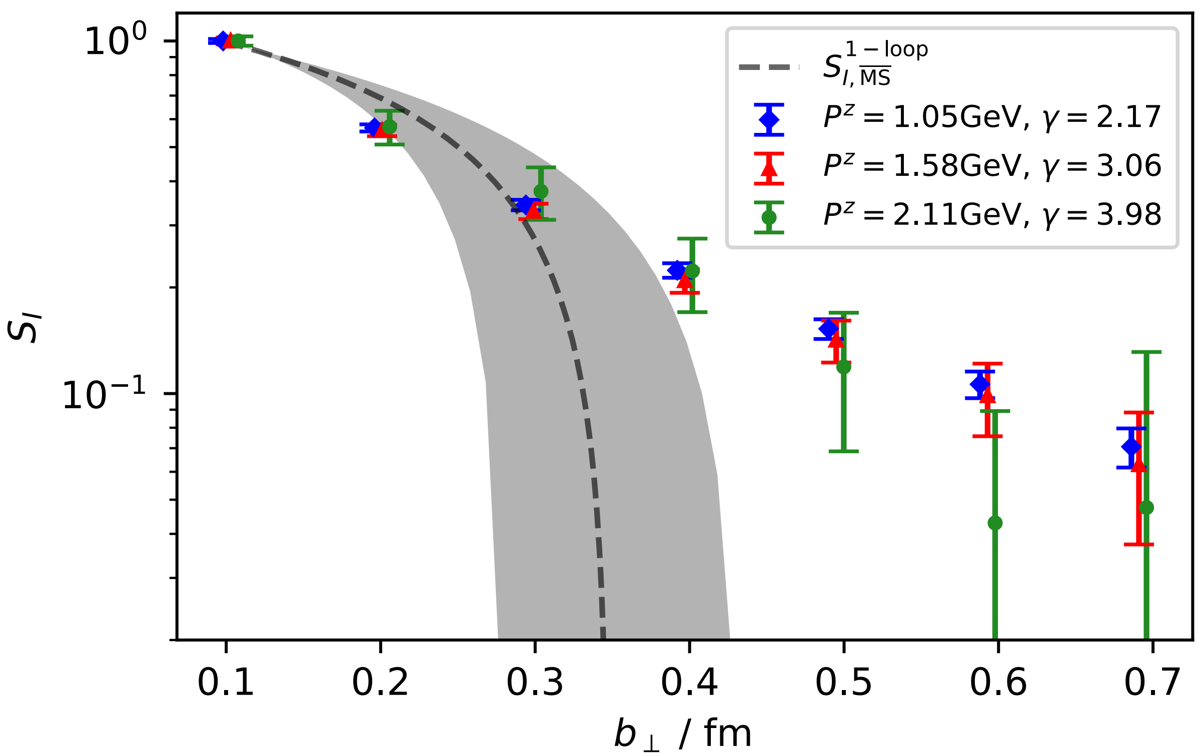
Comparison with phenomenology



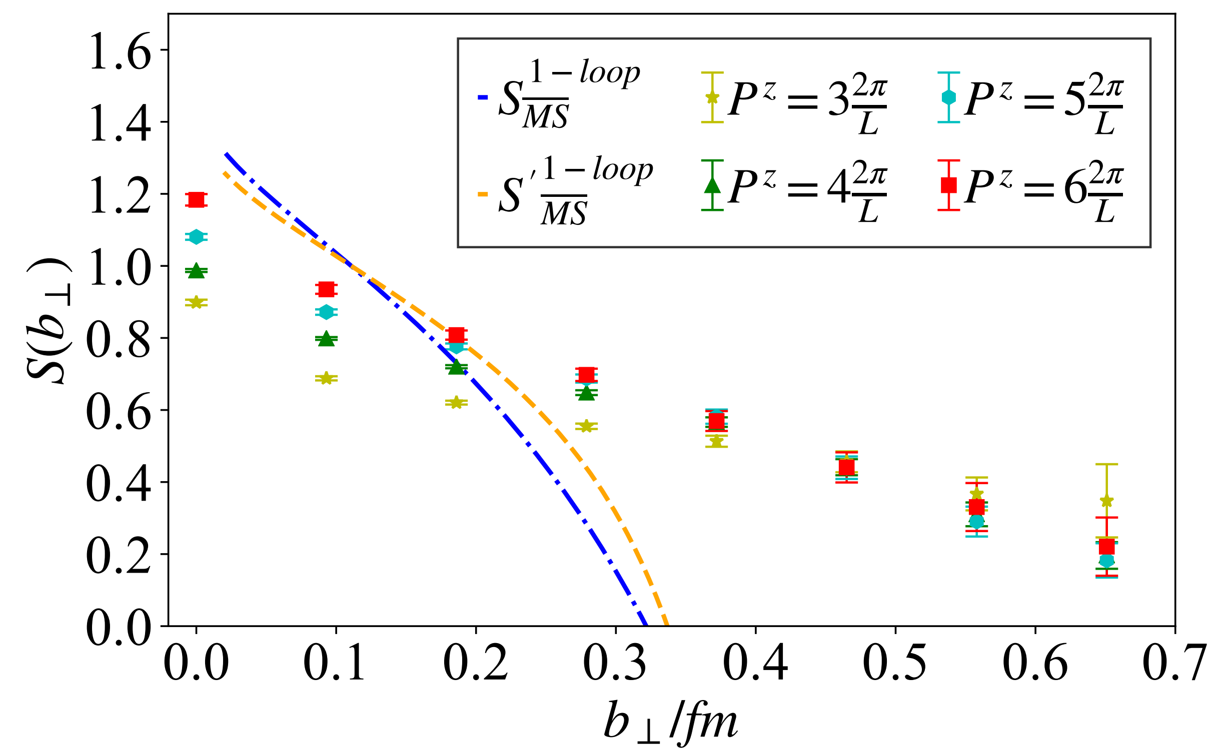
SV19: I. Scimemi and A. Vladimirov, JHEP 06 (2020) 137
Pavia19: A. Bacchetta et al., JHEP 07 (2020) 117

(Green points), P. Shanahan, M. Wagman and YZ, PRD 104 (2021).

Reduced soft function



Q.-A. Zhang, et al. (LPC), PRL 125 (2020).



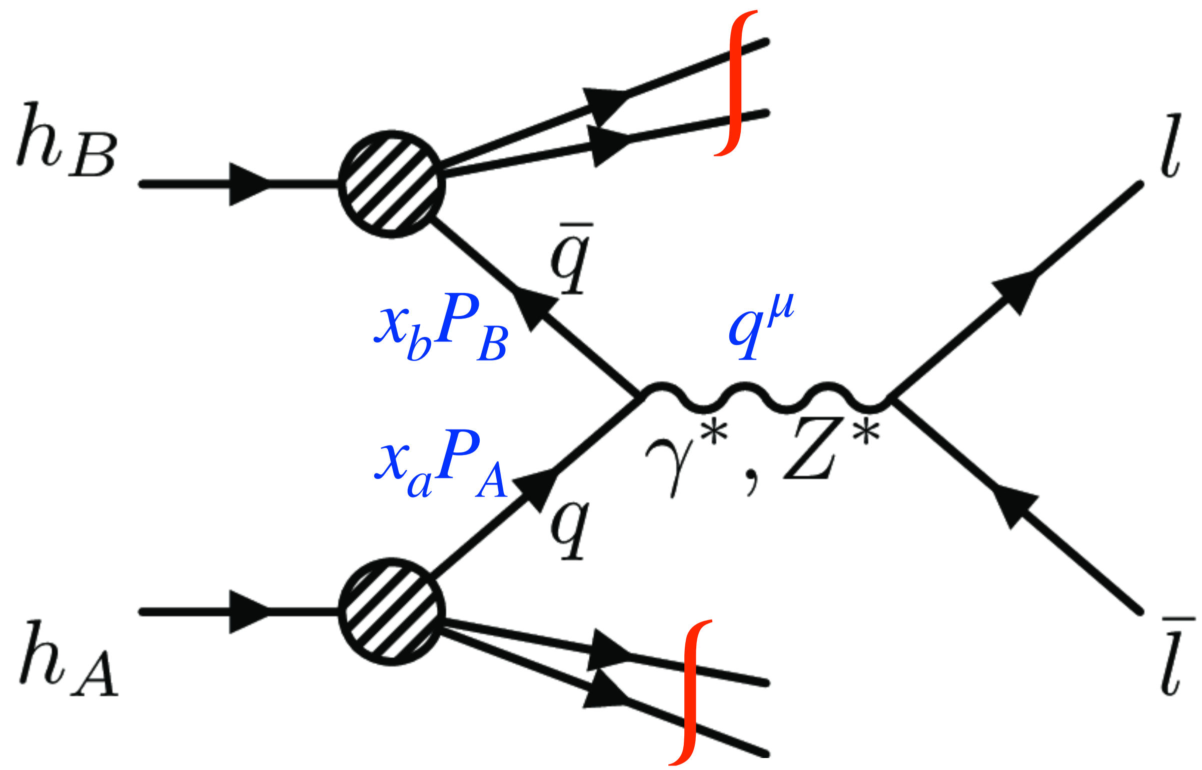
Y. Li et al., PRL 128 (2022).

With the quasi beam function, soft function, and Collins-Soper kernel calculable, we can obtain the full TMD from lattice QCD!

Conclusion

- New large-rapidity (LR) scheme;
- The quasi TMD is equivalent to the LR scheme through Lorentz invariance;
- The LR and Collins schemes differ by the order of UV renormalization and light-cone limits, so we can perturbatively match them;
- We derive the factorization formula for both quark and gluon quasi TMDs using such relations;
- There is no mixing between quarks of different flavors, quark and gluon channels, or different spin structures.

Drell-Yan production of lepton pair



$$n_a = (1,0,0,1)/\sqrt{2}\,, \qquad n_b = (1,0,0,-1)/\sqrt{2}$$

$$q^\mu = l^\mu + \bar{l}^\mu\,, \qquad Q^2 = q^2\,, \qquad Y = \frac{1}{2}\ln\frac{n_a\cdot q}{n_b\cdot q}$$

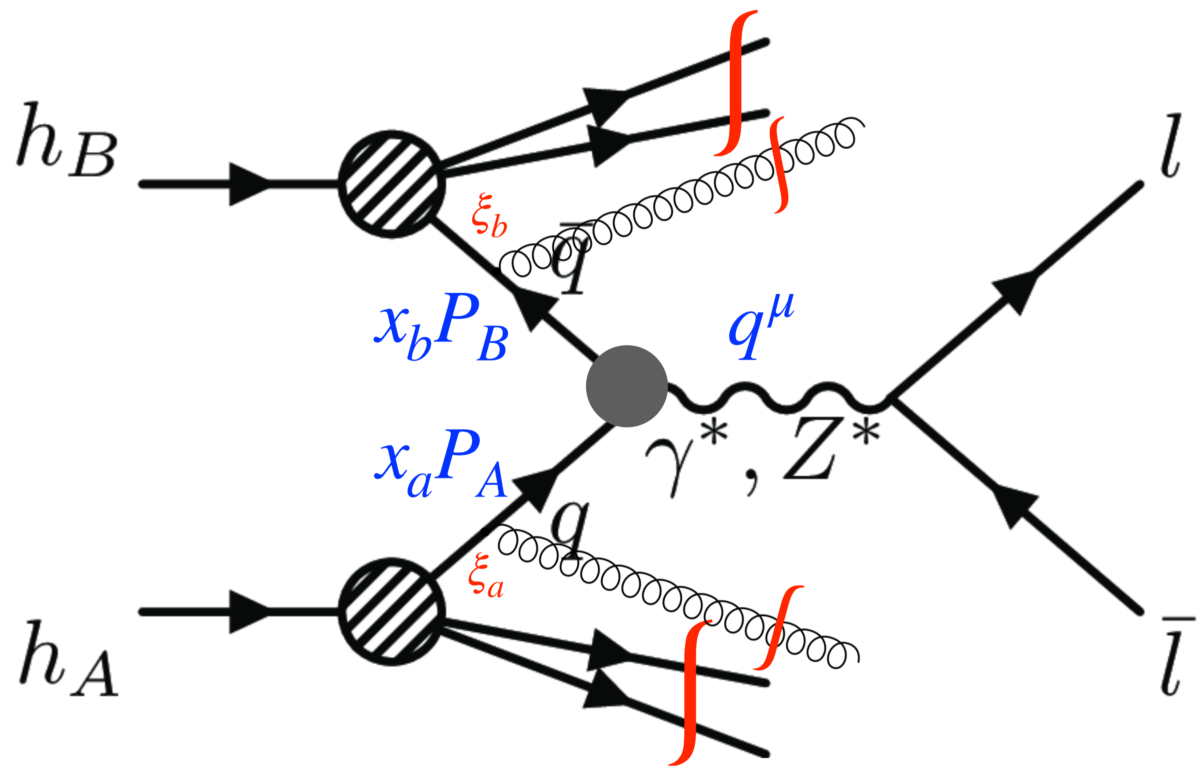
$$s = (P_A + P_B)^2$$

$$x_a = Qe^{+Y}\sqrt{s}, \quad x_b = Qe^{-Y}\sqrt{s}$$

Collinear factorization:

$$\frac{d\sigma_{\rm DY}}{dQ^2 dY} = \sum_{i,j}\int_{x_a}^{1} d\xi_a \int_{x_b}^{1} d\xi_b\; f_{i/h_A}(\xi_a)\; f_{i/h_A}(\xi_b)\frac{d\hat{\sigma}_{ij}(\xi_a,\xi_b)}{dQ^2 dY}\left[1 + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{Q^2}\right)\right]$$

Drell-Yan production of lepton pair



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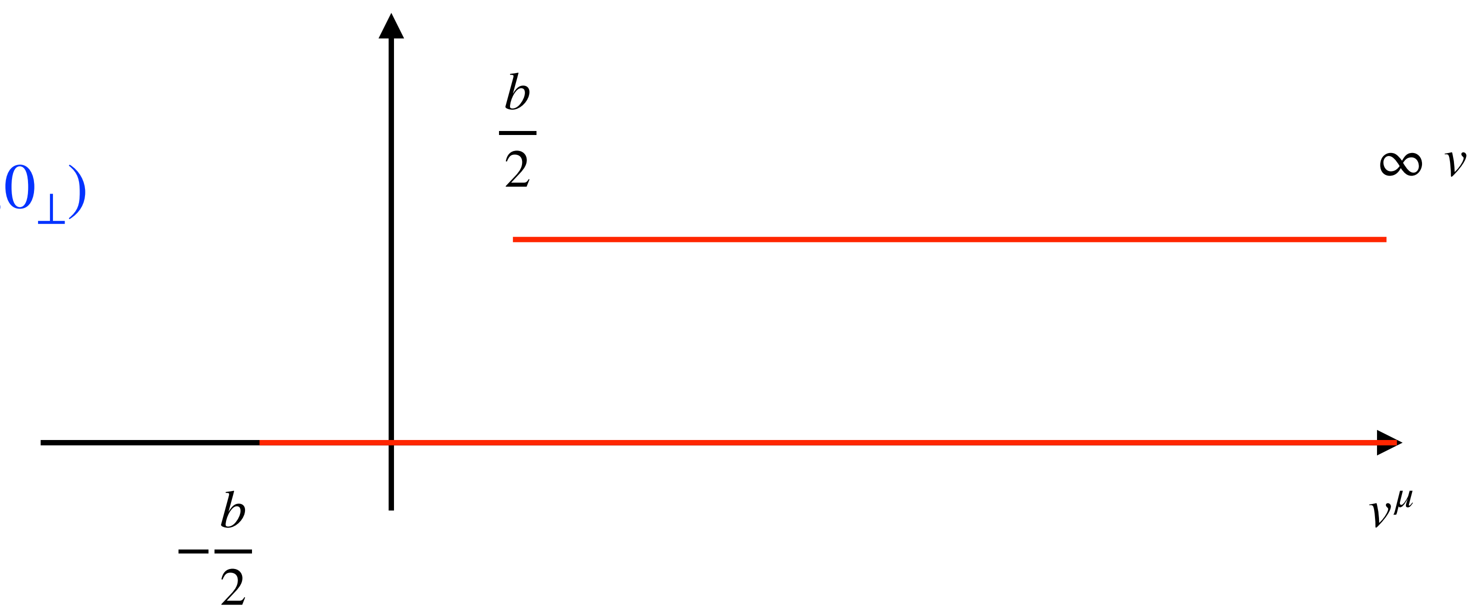
Ji-Ma-Yuan (JMY) scheme

$b^\mu = (0, b^-, b_\perp)$ $\delta^\mu = (0, b^-, 0_\perp)$

$v^\mu = (v^+, v^-, 0_\perp)\,,\quad v^- \gg v^+ > 0$

$\tilde{v}^\mu = (\tilde{v}^+, \tilde{v}^-, 0_\perp)\,,\quad \tilde{v}^+ \gg \tilde{v}^- > 0$

$|\eta| \to \infty$



$$B^{\text{JMY}}_{q/h}(x, \vec{b}_T, \mu, \zeta_v) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \Phi^{[\gamma^+]}_{q/h}[b, P, \mu, -\infty v, b^- n_b]$$

$$S^R_{\text{JMY}}(b_T, \mu, y_A, y_B) = S^R[b_\perp, \mu, -\infty v, -\infty \tilde{v}]$$

TMDPDF:
$$f^{\text{JMY}}_{i/h}(x, \vec{b}_T, \mu, \zeta_v, \rho) = \frac{B^{\text{JMY}}_{i/h}(x, b_T, \mu, \zeta_v)}{\sqrt{S^R_{\text{JMY}}(b_T, \mu, \rho)}}$$

$$\rho^2 = \frac{v^- \tilde{v}^+}{v^+ \tilde{v}^-}$$

$$\zeta_v^2 = \frac{(2P \cdot v)^2}{v^2} = 2(P^+)^2 \frac{v^-}{v^+}$$

Analytically continuable from the LR scheme (y_P-y_B to ϱ).