Dispersive analysis of the form factors in semi-leptonic decays

De-Liang Yao

In collaboration with M. Albaladejo, P. Fernandez-Soler, F.-K. Guo and J. Nieves

Hunan University, Changsha, China

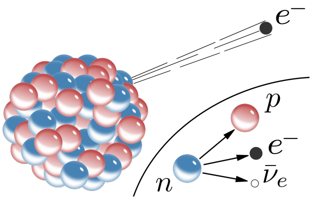
Remote theory seminar @ Jefferson Lab, Jan 24, 2022

Outline

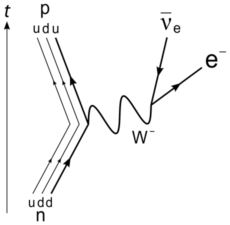
Introduction

Exclusive semi-leptonic decay

❑ Open the era of weak-interaction physics



nuclear(nucleon) β decay



Intermediated W boson

❑ Still significant for modern physics

☞ Accessible both by experiments and lattice QCD

☞ A sharp probe of new physics beyond Standard Model

- Cabibbo-Kobayashi-Maskawa (CKM) matrix elements $\rightarrow$ testing unitarity
- $\mathcal{R}(D^{(*)}) = \frac{\mathcal{B}_{\rightarrow D^{(*)}\tau\nu}}{\mathcal{B}_{\rightarrow D^{(*)}\mu\nu}}$ $\rightarrow$ testing lepton universality

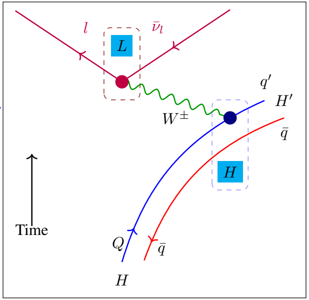
High-precision measurements $\longleftrightarrow$ Accurate theoretical inputs

Exclusive semi-leptonic decay: $H \to H'\ell\bar{\nu}_\ell$

❑ The $H \to H'\ell\bar{\nu}$ decay

☞ Invariant amplitude:

$$\mathcal{M} = \frac{G_F V_{q'Q}}{\sqrt{2}} \underbrace{\left\{\bar{u}(p_\ell)\gamma_\mu(1-\gamma_5)v(p_\nu)\right\}}_{\mathcal{L}_\mu} \underbrace{\left\{\left\langle H'(p')|\bar{q}'\gamma^\mu(1-\gamma_5)Q|H(p)\right\rangle\right\}}_{\mathcal{H}^\mu}$$



☞ Hadronic part:

$$\mathcal{H}^\mu = f_+(q^2)\left[(p+p')^\mu - \frac{m_H^2 - m_{H'}^2}{q^2}q^\mu\right] + f_0(q^2)\frac{m_H^2 - m_{H'}^2}{q^2}q^\mu$$

- Vector FF $f_+(q^2)$: $J^P = 1^-$
- Scalar FF $f_0(q^2)$: $J^P = 0^+$

$$f_+(0) = f_0(0)$$

❑ Form factors (FF)

☞ Necessity

1. Hadrons are non-point-like objects
2. Structure and internal dynamics parametrized in form factors
3. Understanding hadron structure helps understanding QCD

☞ Unknown

1. Non-perturbative objects
2. Nobody knows exact functional form

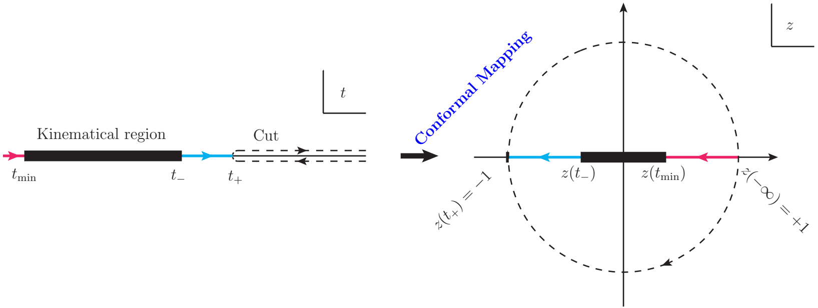
☞ Known

1. Analytic properties
2. Unitarity, etc

Brief overview of traditional parametrizations

z-expansion

- Conformal mapping: map the domain of analyticity onto the **unit circle**



$$z(t,t_0) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}\,, \qquad t_0 = t_+ - \sqrt{t_+ - t_-}\sqrt{t_+ - t_{\min}}$$

- Expand form factors in a Taylor series

$$f(t) = \sum_{n=0}^{\infty} a_n^f(t_0)\, z^n(t,t_0)$$

Boyd-Grinstein-Lebed (BGL) parametrization

❑ BGL approach

[Boyd, Grinstein and Lebed, PRL74(1995)]

$$f(t) = \frac{1}{P(z)\phi(z)} \underbrace{\boxed{\sum_{n=0}^{\infty} a_n^f(t, t_0) z^n(t, t_0)}}_{\text{z expansion}}$$

- ☞ Blaschke factors: $P(z) \longrightarrow$ remove subthreshold poles
- ☞ Outer functions: $\phi(z) \longrightarrow$ unitarity contraints

$\longrightarrow$ Unitarity bounds on the parameters a_n^f: $\sum_{n=0}^{\infty} \left(a_n^f\right)^2 < 1$.

Issue:

Truncation of BGL expansion $\rightarrow$ Unphysical singularity at threshold (t_+) !

❑ BCL approach

$$f(t) = \frac{1}{P(z)\phi(z)} \sum_{n=0}^{N} a_n^f(t, t_0) \left\{ z^n(t, t_0) - (-1)^{n-N-1} \frac{k}{N+1} z^{N+1}(t, t_0) \right\}$$

Caprini-Lellouch-Neubert (CLN) parametrization I

- Symmetries $\longrightarrow$ form factors of $B^{(*)} \to D^{(*)}$ decays share a unique function, the Isgur-Wise function $\xi(\omega)$, in heavy quark limit

Chiral Symmetry: m_u, m_d and $m_s \ll \Lambda_{\rm QCD}$.

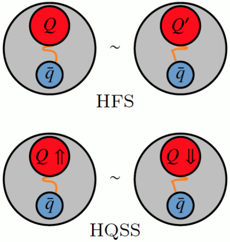
Light pseudoscalar mesons are the Goldstone bosons SSB $SU(3)_L \otimes SU(3)_R \to SU(3)_V$

Heavy quark symmetry (HQS): $H_{\rm QCD}$ $(m_Q \to \infty)$

Flavour does not matter (in $Q\bar{q}$ systems)

$m_H \sim m_Q$, Q static source $\Rightarrow$ dynamics independent of $\vec{S}_Q$

Helps organizing the hadron spectrum



$J = j_l + \frac{1}{2}$

$j_l = \frac{1}{2}$	Exp. mass
$D^+(0^-)$	1869.59
$D^{+*}(1^-)$	2010.26
$D_s^+(0^-)$	1968.28
$D_s^{+*}(1^-)$	2112.10
$D_{s0}^*(0^+)$	2317.70
$D_{s1}^*(1^+)$	2459.50

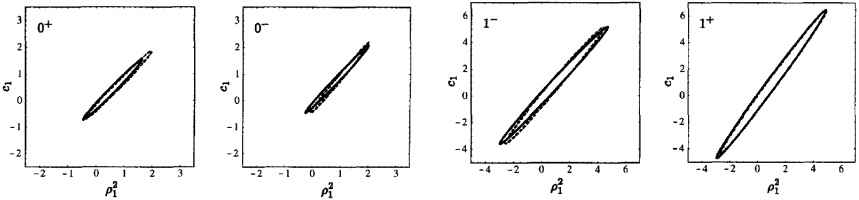
$J = j_l + \frac{1}{2}$

$j_l = \frac{1}{2}$	Exp. mass
$B^-(0^-)$	5279.32
$B^{-*}(1^-)$	5324.65
$\bar{B}_s^0(0^-)$	5366.89
$\bar{B}_s^*(1^-)$	5415.4
$\bar{B}_{s0}^*(0^+)$	?
$\bar{B}_{s1}^*(1^+)$	?

[courtesy of Fernandez-Soler]

Caprini-Lellouch-Neubert (CLN) parametrization II

- Unitarity constraints on the coefficients of $V_1(\omega)$ [Caprini, Lellouch and Neubert, NPB530 (1998)]



- $V_1(\omega)$ is the reference form factor (accuracy better than 2%)

$$V_1(w) = V_0(w_0)\left[1 - \rho_1^2(w - w_0) + c_1(w - w_0)^2 + d_1(w - w_0)^3 + \cdots\right], \quad w = \frac{m_B^2 + m_D^2 - t}{2m_B m_D}$$

- $S_1(w)$ is obtained by the ratio (A, B, C by HQET)

$$\frac{S_1(\omega)}{V_1(\omega)} = A\left[1 + B(\omega - \omega_0) + C(\omega - \omega_0)^2 + \cdots\right],$$

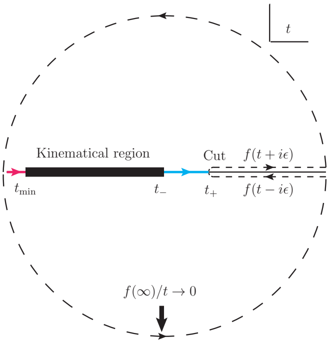
NOTE: $\{V_1, S_1\} \longleftrightarrow \{f_+, f_0\}$

Dispersion representation

❑ Once subtraction

☞ Analyticity:

$$f(t) = f(t_0) + \frac{t - t_0}{\pi} \int_{t_+}^{\infty} \frac{dt'}{t' - t_0} \frac{\mathrm{Im} f(t')}{t' - t}$$



☞ Unitarity:

$$\mathrm{Im}\, f(t) = T^*(t)\rho(t) f(t)$$

❑ Solution

☞ Single channel → analytic Omnès solution

$$f(t) = f(t_0) \exp\left[\frac{t - t_0}{\pi} \int_{t_+}^{\infty} \frac{dt'}{t' - t_0} \frac{\psi(t')}{t' - t - i\epsilon}\right] \Rightarrow$$

a new parametrization for $\bar{B} \to D$

[DLY, Fernandez, Guo, Nieves, PRD(2020)]

☞ Coupled channel → Muskhelishvili-Omnès equation

→ need to be solved numerically. ⇒ **heavy-to-light transitions** [DLY, Albaladejo, Fernandez, Guo, Nieves, EPJC(2018)]

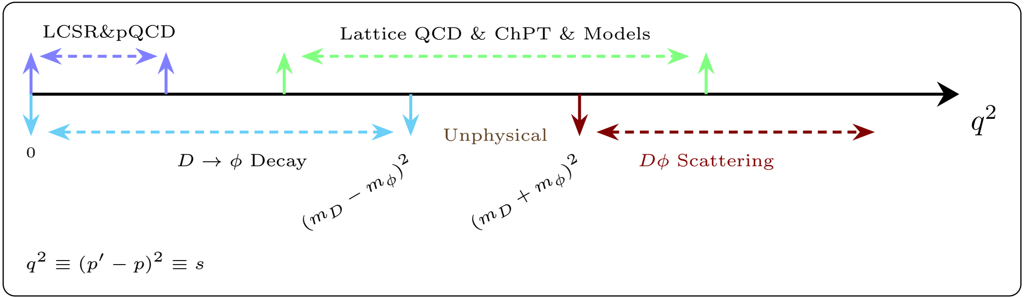
Heavy-to-light scalar form factors

Introduction to $D_{\ell 3}$ decay

Motivation

- Knowledge of $H\phi$ (mainly $D\phi$) interaction accumulated in recent years
- Timely to study the scalar $H \to \phi$ form factor by means of MO representation

Schematic draw of kinematics



✓ Study $f_0(q^2)$ in the whole decay region using **Dispersive techniques**
✓ Incorporate information in scattering region using **Watson Theorem**
} MO formalism

Muskhelishvili-Omnès (MO) formalism

❑ Scalar form factors in coupled channels

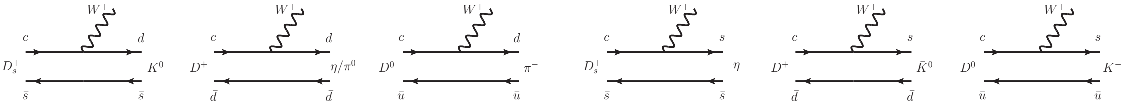
[N.I. Muskhelishvili, Singular integrals equations, (P. Nordhof 1953)] [R. Omnes, Nuovo Cim. 8, 613 (1958)]

✓ $c \to d$ **induced** $(\Delta s = 0, \Delta I = \frac{1}{2})$

$$\begin{pmatrix} \sqrt{\frac{3}{2}} f_0^{D^0\to\pi^-}(s) \\ f_0^{D^+\to\eta}(s) \\ f_0^{D_s^+\to K^0}(s) \end{pmatrix} = \Omega^{(0,\frac{1}{2})}(s)\cdot \vec{\mathcal{P}}^{(0,\frac{1}{2})}(s)$$

✓ $c \to s$ **induced** $(\Delta s = 1, \Delta I = 0)$

$$\begin{pmatrix} -\sqrt{2} f_0^{D^0\to K^-}(s) \\ f_0^{D_s^+\to\eta}(s) \end{pmatrix} = \Omega^{(1,0)}(s)\cdot \vec{\mathcal{P}}^{(1,0)}(s)$$



❑ MO representation

✓ $\vec{\mathcal{F}}(s) = \Omega(s)\cdot\vec{\mathcal{P}}(s)$, $\qquad \mathrm{Im}\vec{\mathcal{F}}(s) = T^*(s)\Sigma(s)\vec{\mathcal{F}}(s)$

✓ $\Omega(s) = \frac{1}{\pi}\int_{s_{\rm th}}^{\infty} \frac{T^*(s)\Sigma(s)\Omega(s)}{s'-s-i\epsilon} ds'$, $\to$ **T** : Dϕ interaction

✓ $\vec{\mathcal{P}}(s) = \vec{\alpha}_0 + \vec{\alpha}_1 s$, (rank one)

$\vec{\mathcal{P}}(s)$: Subtractions constrained by chiral matching

✓ **Tree-Level ChPT**

$$\mathcal{L}_0 = f_D(mD_\mu^* - \partial_\mu D)u^\dagger J^\mu \ ,$$

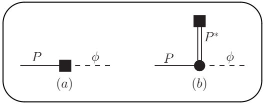
$$\mathcal{L}_1 = \beta_1 Du(\partial_\mu U^\dagger)J^\mu + \beta_2(\partial_\mu\partial_\nu D)u(\partial^\nu U^\dagger)J^\mu$$

$$\mathcal{L}_{DD^*\phi} = i\tilde{g}(D_\mu^* u^\mu D^\dagger - Du^\mu D_\mu^{*\dagger}) \ .$$

Only 2 relevant unknown LECs

Form factors in heavy meson ChPT

Chiral expansion of the form factors



$$f_+^{[P\phi](S,I)}(s) = \frac{\mathcal{C}_{[P\phi]}^{(S,I)}}{\sqrt{2}F_0}\left[\frac{f_P}{\sqrt{2}} + \sqrt{2}\frac{\tilde{g}\, m\, f_P}{m_R^2 - s} + \beta_1^P - \frac{\beta_2^P}{2}(\Sigma_{P\phi} - s)\right],$$

$$f_0^{[P\phi](S,I)}(s) = \frac{\mathcal{C}_{[P\phi]}^{(S,I)}}{\sqrt{2}F_0}\left[\left(\sqrt{2}\frac{\tilde{g}\, m f_P}{m_R^2} + \beta_1^P\right)\frac{\Delta_{P\phi} - s}{\Delta_{P\phi}} + \left(\sqrt{2}f_P - \beta_2^P(\Sigma_{P\phi} - s)\right)\frac{\Delta_{P\phi} + s}{2\,\Delta_{P\phi}}\right].$$

- Abbreviations:
  $\Delta_{P\phi} = m_P^2 - m_\phi^2,\ \Sigma_{P\phi} = m_P^2 + m_\phi^2$
- Scaling properties:
  $\beta_1^P \sim \sqrt{m_P},\ \beta_2^P \sim 1/\sqrt{m_P^3}$

- Coefficients

(S,I) channel	$(1,0)$ DK	$D_s\eta$	$(0,\frac{1}{2})$ $D\pi$ $\bar{B}\pi$	$D\eta$ $\bar{B}\eta$	$D_s\bar{K}$ $\bar{B}_s\bar{K}$
$\mathcal{C}$	$-\sqrt{2}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{6}}$	1

Mathching (s_0: matching point)

$$\Omega(s_0)\cdot\vec{\mathcal{P}}(s_0) = \vec{\mathcal{F}}_\chi(s_0) \quad\Longrightarrow\quad \begin{cases} \vec{\alpha}_0 = \Omega^{-1}(s_0)\cdot\vec{\mathcal{F}}_\chi(s_0) - \vec{\alpha}_1\, s_0\,, \\ \vec{\alpha}_1 = \Omega^{-1}(s_0)\cdot\left[\vec{\mathcal{F}}'_\chi(s_0) - \Omega'(s_0)\cdot\Omega^{-1}(s_0)\cdot\vec{\mathcal{F}}_\chi(s_0)\right]. \end{cases}$$

$D\phi$ interaction in UChPT

❑ Chiral potentials from covariant ChPT & Unitarization

- ☞ NLO: [Kolomeitsev & Lutz, PLB582(2004)39] [Guo et al, PLB641(2006)278] [Liu et al, PRD87(2013)014508] [Altenbuchinger et al, PRD89(2014)014026]...
- ☞ NNLO: [Geng et al, PRD82, 054022 (2010)] [Du, Guo, Meissner & Yao, JHEP11(2015)058 & EPJC77(2017)728]

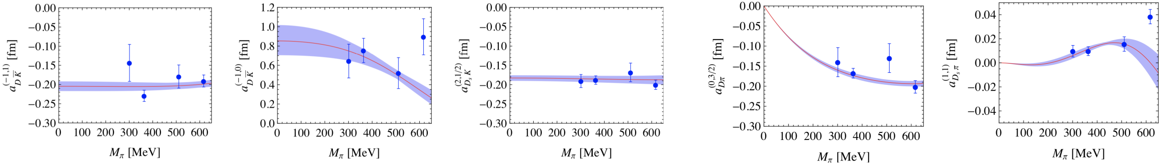
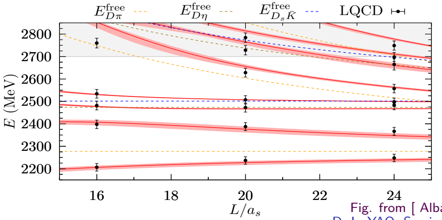


Fig. from [Liu et al, PRD87(2013)014508]

❑ Applications of NLO potentials by Liu et al

Energy Level $(S, I) = (0, 1/2)$



Two-pole structure of $D^*(2300)$

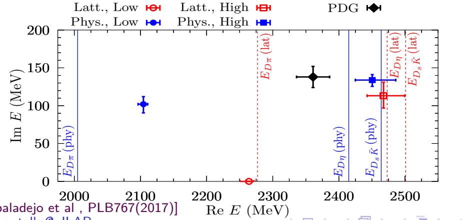


Fig. from [Albaladejo et al , PLB767(2017)]

MO problem: I. Inputs

- T-matrix

Threshold — **Matching point** — **S-Matrix parameterization:** — **Asymptotic Condition:**

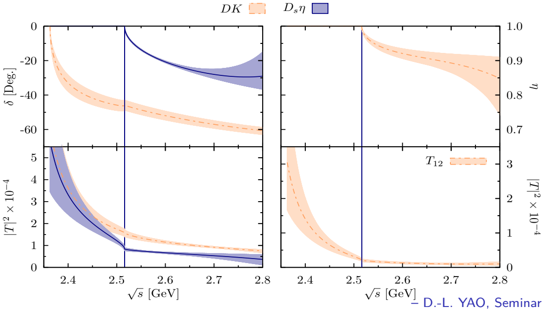
$S_{\rm thr}$ — ChPT & Unitarization — s_m — ∞

$$T^U = \frac{V}{1 - VG}$$

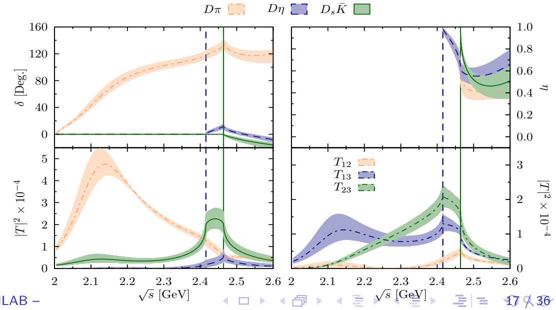
$$T^H \Leftarrow \begin{array}{l} S_{ii}(s) = \eta_j e^{2i\delta_i} \quad S_{ij}(s) = \gamma_{ij} e^{i\phi_{ij}} \\ S_{ij}(s) = \delta_{ij} + 2i\sigma_i^{1/2}\sigma_j^{1/2} T_{ij}(s) \end{array}$$

$$\begin{array}{l} X_i(s) = X_i(\infty) + [X_i(s_m) - X_i(\infty)]\frac{2}{1+(s/s_m)^{3/2}} \\ X \in \{\delta,\ \eta\} \quad \delta_1(\infty) = n\pi, \quad \delta_{i\neq 1}(\infty) = 0 \end{array}$$

- $(S, I) = (1, 0)$
  - ☞ $s_m = (2.7\ \text{GeV})^2$

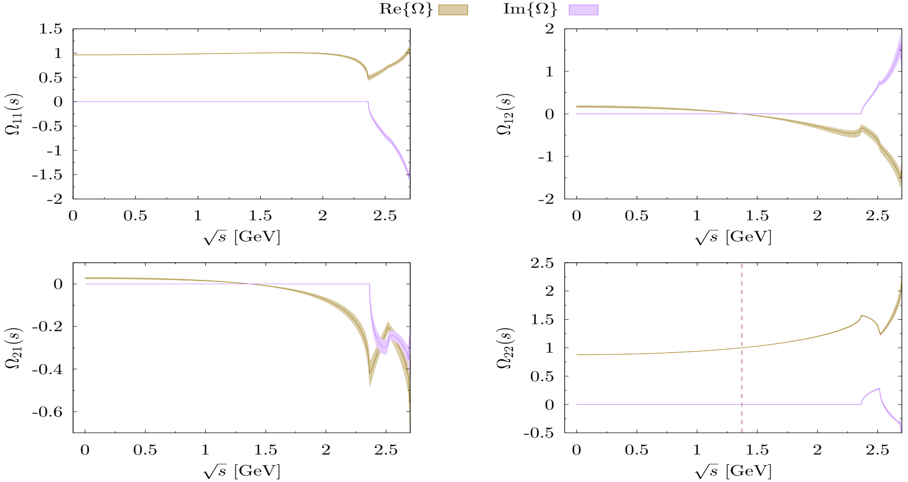


- $(S, I) = (0, \frac{1}{2})$
  - ☞ $s_m = (2.6\ \text{GeV})^2$



MO problem: II. Solutions

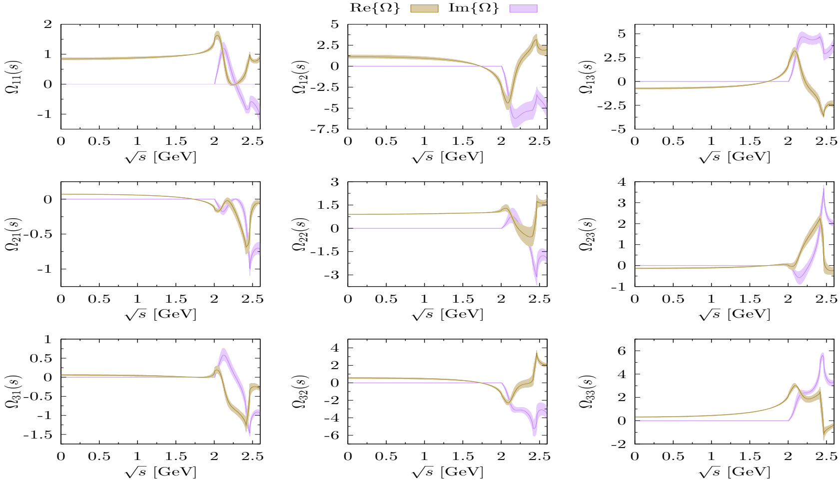
❑ MO-matrix for channel $(S, I) = (1, 0)$ ✓ Normalised at $s = 0$



Error propagated from the uncertainties of LECs in chiral potential

MO problem: II. Solutions

❑ MO-matrix for channel $(S, I) = (0, \frac{1}{2})$ ✓ Normalised at $s = 0$

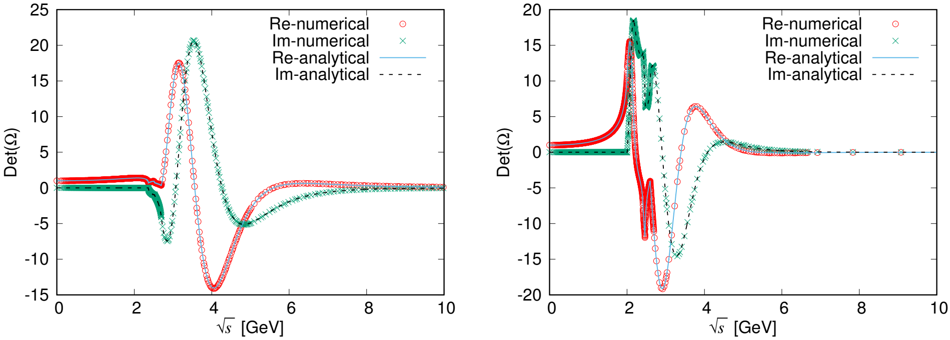


Error propagated from the uncertainties of LECs in chiral potential

MO problem: III. Checking

- Determinant of the MO matrix also satisfies an Omnès-type dispersion relation

$$\det \Omega(s) = \exp\left[\frac{s}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\psi(s')}{s'(s'-s-i\epsilon)}{\rm d}s'\right]\ , \quad \exp(2i\psi(s)) \equiv \det[\mathbb{1} + 2i\mathbf{T}(s)\Sigma(s)]$$



Analytical v.s. Numerical

Fit to lattice QCD data

- Latest data $(8+8)$ with hypercubic effects by ETM [PRD96(2017)054514]

$$\begin{pmatrix} \sqrt{\frac{3}{2}} f_0^{D^0\to\pi^-}(s) \\ f_0^{D^+\to\eta}(s) \\ f_0^{D_s^+\to K^0}(s) \end{pmatrix} = \Omega^{(0,\frac{1}{2})}(s)\cdot \vec{\mathcal{P}}^{(0,\frac{1}{2})}(s) \qquad \begin{pmatrix} -\sqrt{2} f_0^{D^0\to K^-}(s) \\ f_0^{D_s^+\to\eta}(s) \end{pmatrix} = \Omega^{(1,0)}(s)\cdot \vec{\mathcal{P}}^{(1,0)}(s)$$

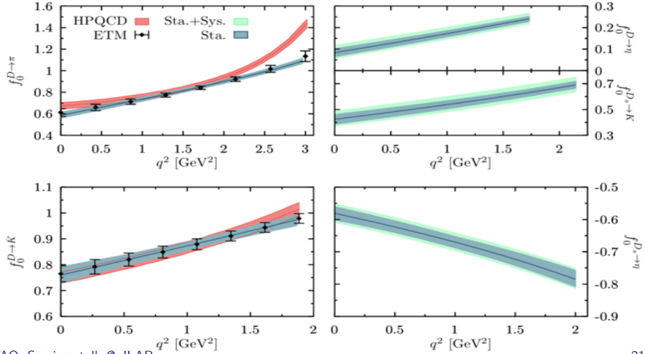
- Contribution of $D_{s0}^*(2317)$ below threshold

$$\vec{\mathcal{P}}^{(1,0)}(s) \to \frac{\beta_0 \vec{\Gamma}}{s-s_p} + \vec{\mathcal{P}}^{(1,0)}(s)$$

Results

	Value	correlation matrix			
χ^2	1.82	β_1	β_2	β_0	δ_χ
β_1	0.08(2)	1	0.96	−0.03	0.68
β_2	0.07(1)		1	0.09	0.84
β_0	0.12(1)			1	0.11
δ_χ	−0.21(2)				1

Hypercubic effects should be taken into account in lattice simulation!



HPQCD:
[Na, et al, PRD82(2010)114506] [Na, et al, PRD84(2011)114505]

Extended to bottom sector: combined fit

- Heavy Quark Flavor Symmetry (HQFS)
  - ✓ Formalism applies to bottom sector straightforwardly
  - ✓ However, LECs need to be adjusted

$b \to u$ induced

$$\begin{pmatrix} \sqrt{\frac{3}{2}} f_0^{\bar{B}^0 \to \pi^+}(s) \\ f_0^{B^- \to \eta}(s) \\ f_0^{\bar{B}_s^0 \to K^+}(s) \end{pmatrix} = \Omega_{\bar{B}}^{(0,\frac{1}{2})}(s) \cdot \vec{\mathcal{P}}_{\bar{B}}^{(0,\frac{1}{2})}(s)$$

- Scaling

$[D\phi]$ to $[\bar{B}\phi]$ Interactions	$[D \to \phi]$ to $[\bar{B} \to \phi]$ Decays
$h_{0,1,2,3} \sim m_Q \;,\; h_{4,5} \sim \frac{1}{m_Q}$	$\beta_1 \sim m_Q^{\frac{1}{2}} \;,\quad \beta_2 \sim m_Q^{-\frac{3}{2}}$
$h^B_{0,1,2,3} \sim \frac{\bar{m}_B}{\bar{m}_D} h^D_{0,1,2,3} \;,\quad h^B_{4,5} \sim \frac{\bar{m}_D}{\bar{m}_B} h^D_{4,5}$	$\beta_1^D = \sqrt{\frac{\bar{m}_D}{\bar{m}_B}} \beta_1^B (1+\delta) \;,\quad \beta_2^D = \sqrt{\frac{\bar{m}_B^3}{\bar{m}_D^3}} \beta_2^B (1-3\delta)$

- Combined fit: lattice data from both D and B sectors & LCSR results

$$\chi^2 = (\chi^2)^{\bar{B}\to\pi} + (\chi^2)^{\bar{B}_s\to K} + (\chi^2)^{\bar{B}\to\pi}_{\rm LCSR} + (\chi^2)^{\bar{B}_s\to K}_{\rm LCSR} + (\chi^2_{\rm cov})^{D\to\pi} + (\chi^2_{\rm cov})^{D\to\bar{K}}$$

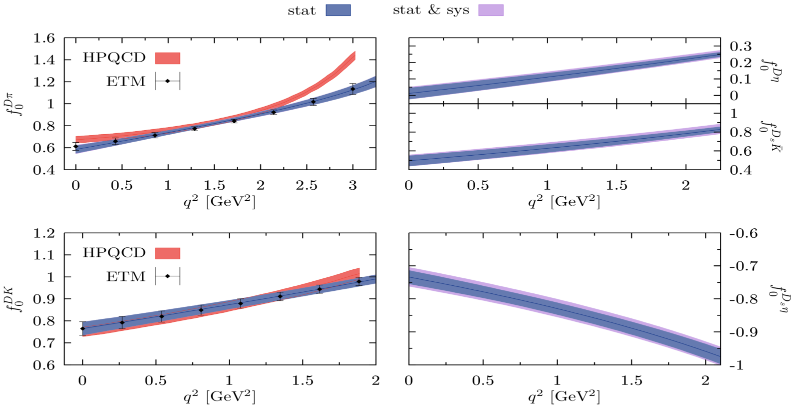
UKQCD&FL-MILC&HPQCD **LCSR** **ETM**

[Flynn et al, PRD91, Barley et al, PRD92] [Bharucha, JHEP1205(2012)092] [Lubicz et al, PRD96(2017)054514]

[Bouchard et al, PRD90(2014)054506] [Duplancic et al, JHEP0804(2008)]

Extended to bottom sector: combined fit

❑ Charm Sector



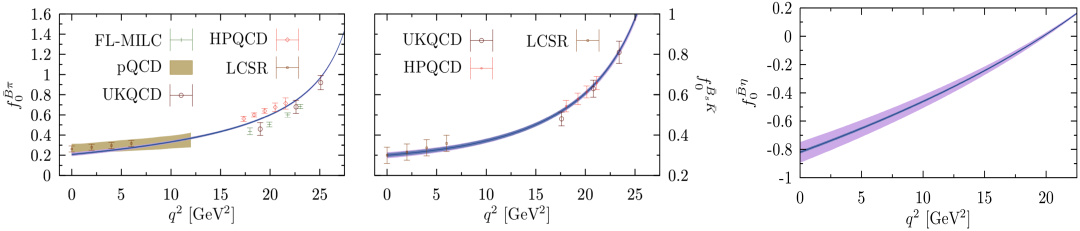
☞ Here almost the same results are obtained as the fit only to data of charm sector

$$f_+^{D\to\pi}(0)|V_{cd}| = 0.1426(19) \qquad \xrightarrow{\text{CKM}} \qquad |V_{cd}| = 0.244(22)$$

$$f_+^{D\to\bar{K}}(0)|V_{cs}| = 0.7226(34) \qquad\qquad |V_{cs}| = 0.945(41)$$

Extended to bottom sector: combined fit

❑ Bottom Sector



Results

$\frac{\chi^2}{\mathrm{dof}} \simeq 2.77$		
β_0	0.152(14)(13)	$f_+^{\bar{B}\to\pi}(0)\lvert V_{ub}\rvert = (8.9 \pm 0.3) \times 10^{-4}$
β_1^B	0.22(4)(4)	
β_2^B	0.0346(16)(15)	**CKM** $\downarrow$
δ	0.138(21)(18)	
δ'	$-0.18(4)(2)$	$10^3\lvert V_{ub}\rvert = 4.3(7)$

- ☞ Lattice QCD data in both sectors are well described simultaneously.
- ☞ Scalar FFs are predicted in the whole kinematical region.

FL+MILC: [Bailey et al, PRD92(2015)014024] pQCD: [Li et al, PRD85(2012)074004] [Wang et al, PRD86(2012)114025]

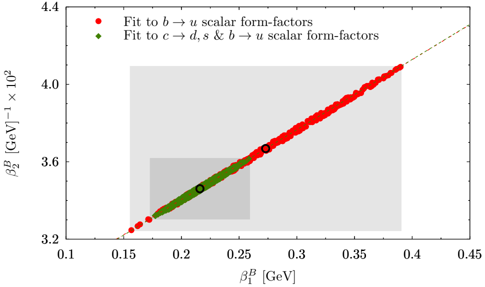
HPQCD: [Dalgic et al, PRD73(2006)074502] [Bouchard et al, PRD90(2014)054506]

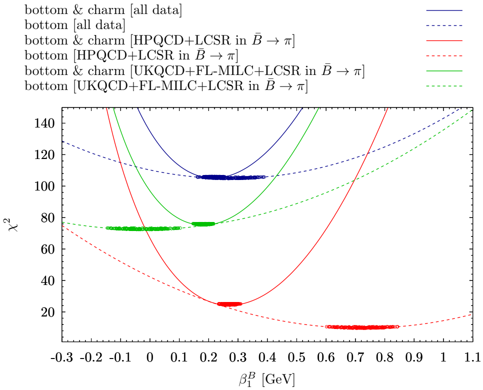
Extended to bottom sector: combined fit

- Assessment of the fit
  - Left: Correlation between β_1 and β_2

$$\frac{\beta_1^B - \bar{\beta}_1^B}{\sigma_{\beta_1^B}} = \frac{\beta_2^B - \bar{\beta}_2^B}{\sigma_{\beta_2^B}}$$

  - Right: Dependence of χ^2 on β_1

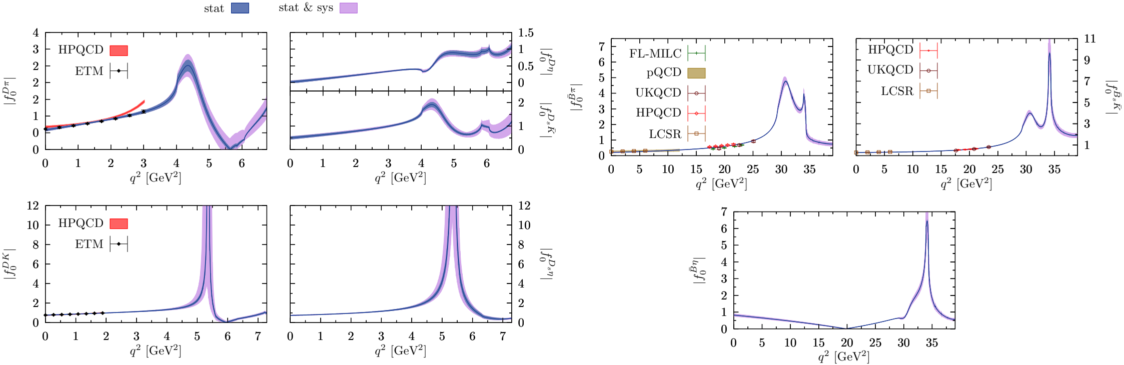




The combined fit provides large curvatures of χ^2, leading to better determination of β_1

Extended to bottom sector: combined fit

❑ Scalar form factors above the $q^2_{\rm max}$ -region



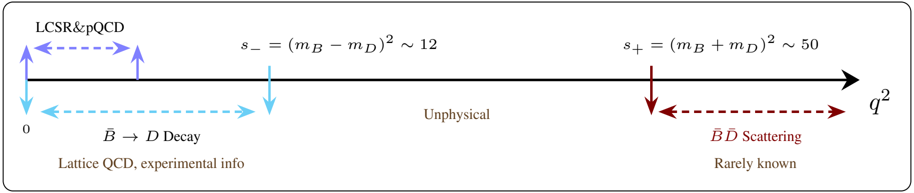
☞ Some signatures of the two-pole structure of $D_0^*(2300)$: 2.1 & 2.45 GeV

☞ The sensitivity of the form factors to the details of the two resonances is not dramatic .

A new parametrization for $\bar{B} \to D$ transition

A new parametrization for $\bar{B} \to D$ transitions

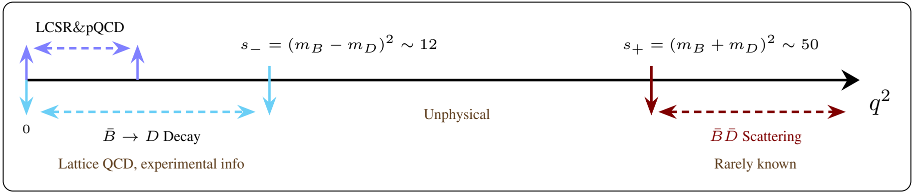
- Tensions
  - inclusive & exclusive
    1. HFLAV average $|V_{cb}|_{\rm in} = (42.19 \pm 0.78) \times 10^{-3}$. [Amhis et al, EPJC77(2017)]
    2. Belle determination $|V_{cb}|_{\rm ex,CLN} = (39.86 \pm 1.33) \times 10^{-3}$. [Glattauer et al, PRD93(2016)]
  - theoretical v.s. experimental
    1. FL-MILC: $\mathcal{R}_D = 0.299(11)$; HPQCD: $\mathcal{R}_D = 0.300(8)$. [Bailey et al, PRD92(2015); Na et al, PRD92(2015)]
    2. BaBar: $\mathcal{R}_D = 0.440(58)(42)$; Belle: $\mathcal{R}_D = 0.375(64)(26)$. [Lees et al, PRL109(2012); Hushcle et al, PRD92(2015)]
- Kinematics for $\bar{B} \to D\ell\bar{\nu}_\ell$ decay



✓ Study FF in the whole decay region using **Dispersive techniques**
✓ Transfer information from decay region to faraway scattering region
} towards a new parametrization

A new parametrization for $\bar{B} \to D$ transitions

❑ $\bar{B} \to D$ form factors



☞ Dispersive representation

$$f(s) = f(s_0) \exp\left[\frac{s-s_0}{\pi}\int_{s_+}^{\infty}\frac{\mathrm{d}s'}{s'-s_0}\frac{\alpha(s')}{s'-s-i\epsilon}\right]$$

$\alpha(s')$: phase of form factor near threshold

$=$ phase shift (elastic region)

☞ A new parametrization $(s' \geq s_+ \gg s_- \geq s_0)$

$$f(s) = f(s_0)\prod_{n=0}^{\infty} \exp\left[\frac{s-s_0}{s_+}\mathcal{A}_n(s_0)\frac{s^n}{s_+^n}\right]$$

$$\mathcal{A}_n(s_0) \equiv \frac{1}{\pi}\int_{s_+}^{\infty}\frac{\mathrm{d}s'}{s'-s_0}\frac{\alpha(s')}{(s'/s_0)^{n+1}}$$

The coefficients $\mathcal{A}_n$ are called Phase Moments.

physically meaningful $\to$ important information on $\bar{B}\bar{D}$ interations

A new parametrization for $\bar{B} \to D$ transitions

❑ Description of lattice QCD and experimental data [Phys.Rev.D **101** (2020), 034014]

☞ The new parametrization

$$f(s) = f(s_0) \prod_{n=0}^{\infty} \exp\left[\frac{s-s_0}{s_+} \mathcal{A}_n(s_0) \frac{s^n}{s_+^n}\right]$$

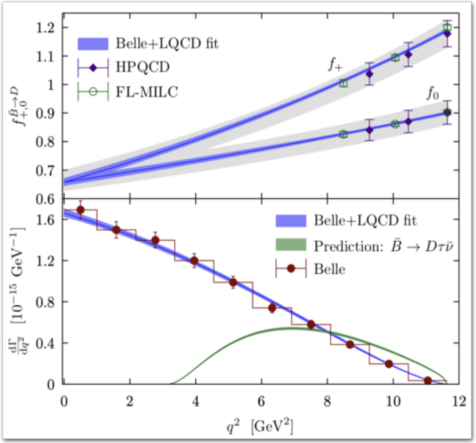
$$\mathcal{A}_n(s_0) \equiv \frac{1}{\pi} \int_{s_+}^{\infty} \frac{\mathrm{d}s'}{s'-s_0} \frac{\alpha(s')}{(s'/s_0)^{n+1}}$$

☞ Fit results

$\frac{\chi^2}{\text{dof}} = \frac{6.47}{22-4} \simeq 0.36$	
$f_0(0)$	0.658(17)
$\mathcal{A}_0^0$	1.38(12)
$\mathcal{A}_0^+$	2.60(12)
$\lvert V_{cb}\rvert \times 10^3$	41.01(75)

$$\mathcal{R}_D = \frac{\mathcal{BR}(\bar{B} \to D\tau\bar{\nu}_\tau)}{\mathcal{BR}(\bar{B} \to D\ell\bar{\nu}_\ell)} = 0.301(5)$$

- $n = 0$ is sufficient
- Independent of subtraction point s_0: $s_0 = 0$



Results: the $|V_{cb}|$ CKM matrix element

❑ Comparison to other determinations

$\lvert V_{cb}\rvert \times 10^3$	Method	Decay mode	Source
41.01 ± 0.75	Phase moments	$\bar{B} \to D\ell\bar{\nu}_\ell$	This work
39.18 ± 1.01	CLN	$\bar{B} \to D\bar{\ell}\nu_\ell$	HFLAV 2017
39.05 ± 0.75	CLN	$\bar{B} \to D^*\bar{\ell}\nu_\ell$	
42.19 ± 0.78	Kinetic scheme	Inclusive	
41.98 ± 0.45	$1S$ scheme	Inclusive	
39.86 ± 1.33	CLN	$B \to \bar{D}\bar{\ell}\nu_\ell$	Belle [Glattauer:2015teq]
40.83 ± 1.13	BGL		
38.4 ± 0.87	CLN	$B^0 \to D^{*-}\ell^+\bar{\nu}_\ell$	Belle [Abdesselam:2018nnh]
42.5 ± 0.97	BGL		
40.49 ± 0.97	BGL	$\bar{B} \to D\ell\bar{\nu}_\ell$	[Bigi and Gambino, PRD94(2016)]
38.2 ± 1.4	CLN	$\bar{B} \to D^*\ell\bar{\nu}_\ell$	[Bigi, Gambino and Schacht, PLB769(2017)]
$40.4^{+1.6}_{-1.7}$	BGL		
$41.9^{+2.0}_{-1.9}$	BGL	$\bar{B} \to D^*\ell\bar{\nu}_\ell$	[Grinstein and Kobach, PLB771(2017)]

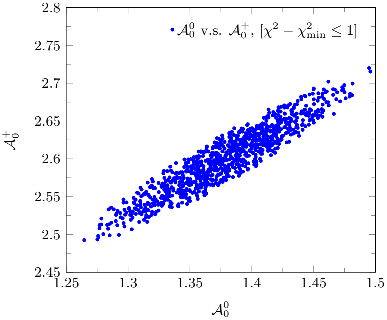
Tension between the exclusive and inclusive determinations was due to the use of the CLN parametrization

Results: the phase moments

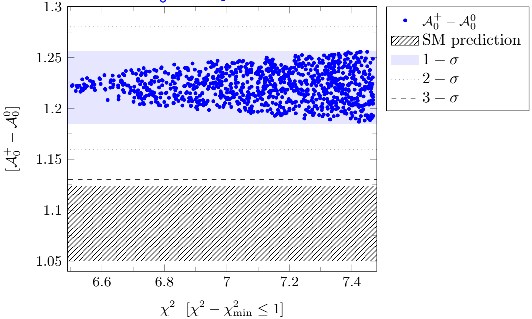
- HQET determination ($r = m_D/m_B$, ω_0 expanding point). [1.13 ∼ 1.28@ NLO]

$$[\mathcal{A}_0^+ - \mathcal{A}_0^0]_{\rm HQET} = -\frac{(r+1)^2}{r^2 - 2\omega_0 r + 1} \ln \frac{2A(1+\omega_0)r}{(r+1)^2} \simeq 1.05 \sim 1.12 \quad .$$

- Our deterimination



$[\mathcal{A}_0^+ - \mathcal{A}_0^0]_{\rm this\ work} = 1.22(3)$



The coefficient $\mathcal{A}$ should be updated by HQEF

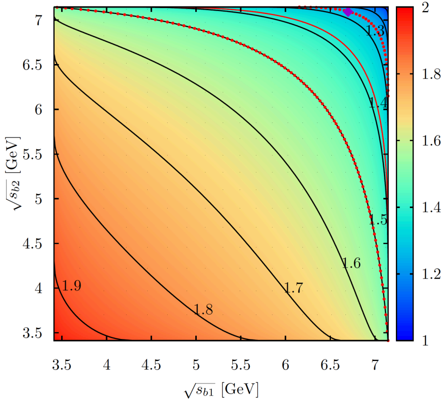
Results: the $\bar{B}\bar{D}$ spectroscopy

- Connecting $\mathcal{A}_0^0$ to $J^P = 0^+$ states below $\bar{B}\bar{D}$ threshold

$$\mathcal{A}_0^0 \equiv \frac{1}{\pi}\int_{t_+}^{\infty} \frac{dt'}{t'-t_0}\frac{\delta(t')}{(t'/t_+)}\ , \quad \delta(t') = \sum_{\sqrt{s_b}\in\mathcal{B}} \left\{\pi - \arctan\left[\frac{\rho(t')t'}{t'-t_-}\sqrt{\frac{s_b-t_-}{t_+-s_b}}\right]\right\}$$

Supporting the existence of two bound states?

$\mathcal{A}_0^0$	$\mathcal{B}$: bound states	
1.38 ± 0.12		
$0.5 \sim 1.0$	$\{s_b \in [t_-, t_+]\}$	
$0.56 \sim 0.54$	$\{\sqrt{s_b} \in [7111, 7133]\}$	[1]
0.68 ± 0.01	$\{\sqrt{s_b} = 6712(18)(7)\}$	[2]
$1.0 \sim 2.0$	$\{\sqrt{s_{b1}}, \sqrt{s_{b2}} \in [t_-, t_+]\}$	
1.26	$\{\sqrt{s_{b1}} = 6699, \sqrt{s_{b2}} = 7094\}$	[3]



[1] Dynamical generated [Sakai et al, PRD96(2017)]
[2] Lattice QCD [Mathur et al, PRL121(2018)]
[3] Quark Model [Mathur Ebert et al, EPJC71(2011)]

Summary and Outlook

1. **Dispersive analysis of form factors**
   - Based on axiomatic principles: Unitarity & Analyticity
   - Elegant bridge connecting heavy-to-light (heavy) decay with heavy-light (heavy) scattering
   - Include coupled-channel effects
2. **Heavy-to-light: successfully describe results of LQCD and LCSR in $D\&B$ sectors**
   - Communicate information between D and B sectors by imposing HQFS
   - Constrain parameters by using chiral symmetry of light quarks
   - Obtain scalar FFs in the fitted channels in the whole kinematical region
   - Predict scalar FFs in the other channels related by chiral symmetry
   - Extract all the heavy-to-light CKM elements
3. **Heavy-to-heavy: a new parametrization for $\bar{B} \to D$ semi-leptonic decay**
   - More efficient than traditional parametrizations like BGL(BCL), CLN approaches:
     - less free parameters & physically meaningful
     - more precise results:

$$|V_{cb}| = (41.01 \pm 0.75) \times 10^{-3} \qquad \& \qquad \mathcal{R}_D = \frac{\mathcal{BR}(\bar{B} \to D\tau\bar{\nu}_\tau)}{\mathcal{BR}(\bar{B} \to D\ell\bar{\nu}_\ell)} = 0.301(5)$$

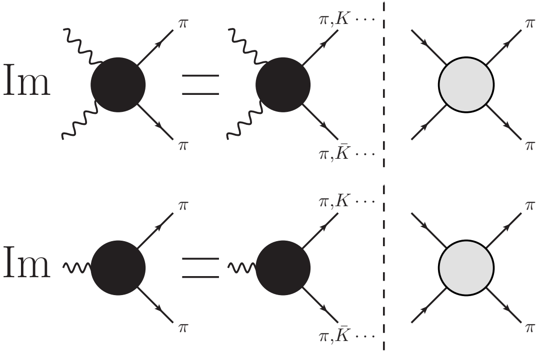
   - Universal for any other semi-leptonic processes induced by $b \to c$ transition:
     - $\bar{B} \to D^*\ell\bar{\nu}$ decay (ongoing)
     - $\bar{\Lambda}_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ decay

Thanks so much for your attention!

Backup

Final state interaction

- "Production" ampl.: weak (spectator); Scattering ampl.: strongly interacting



- "Production"

$$\mathrm{Im} F(s) = T^*(s)\rho(s)F(s)$$

- Scattering

$$\mathrm{Im}\, T(s) = T^*(s)\rho(s)T(s)$$

- Watson's final state interaction theorem: $\phi_F = \phi_T$ (modulo π)
  - Au-Morgan-Pennington method: $F(s) = \alpha(s)T(s)$
  - (Muskhelishvili-)Omnès formalism: $F(s) = P(s)\Omega(s)$

$$\mathrm{Im}\Omega(s) = T^*(s)\rho(s)\Omega(s) \implies \Omega(s) = \exp\left[\frac{s}{\pi}\int_{s_{\rm th}}^{\infty} \mathrm{d}s' \frac{\delta(s')}{s'(s'-s)}\right] .$$

Dispersion relation

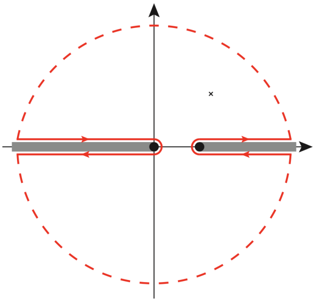
❑ Cauchy's theorem

$$f(s)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(s')}{s'-s}\mathrm{d}s'$$

❑ Dispersion relation

$$f(s)=\frac{1}{\pi}\left[\int_{s_R}^{\infty}+\int_{-\infty}^{s_L}\right]\frac{\mathrm{Im}f(s')}{s'-s-i\epsilon}\mathrm{d}s'$$

☞ $f(s)\to 0$ when $|s|\to\infty$



❑ Dispersion relation with subtractions

$$f(s)=P_{n-1}(s-z_0)+\frac{(s-s_0)^n}{\pi}\int_{\mathrm{cuts}}\mathrm{d}s'\frac{\mathrm{Im}f(s')}{(s'-s_0)^n(s'-s-i\epsilon)}$$

☞ $f(s)/s^n\to 0$ when $|s|\to\infty$;
☞ $\mathrm{Im}f(s_0)=0$.

Parametrization of S matrix

- Two channels

$$T(s)=\begin{pmatrix}\frac{\eta(s)e^{2i\delta_1}-1}{2i\sigma_1(s)} & \frac{\sqrt{1-\eta^2}e^{i\phi_{12}}}{2\sqrt{\sigma_1(s)\sigma_2(s)}}\\ \frac{\sqrt{1-\eta^2}e^{i\phi_{12}}}{2\sqrt{\sigma_1(s)\sigma_2(s)}} & \frac{\eta(s)e^{2i\delta_2}-1}{2i\sigma_2(s)}\end{pmatrix}\,,\quad \phi_{12}=\delta_1+\delta_2+\mathrm{mod}(\pi),\quad 0\leq\eta\leq 1$$

- Three channels

$$S(s)=\begin{pmatrix}\eta_1 e^{2i\delta_1} & \gamma_{12}e^{i\phi_{12}} & \gamma_{13}e^{i\phi_{13}}\\ \gamma_{12}e^{i\phi_{12}} & \eta_2 e^{2i\delta_2} & \gamma_{23}e^{i\phi_{23}}\\ \gamma_{13}e^{i\phi_{13}} & \gamma_{23}e^{i\phi_{23}} & \eta_3 e^{2i\delta_3}\end{pmatrix}\,,\quad \begin{cases}\gamma_{ij}^2=\frac{1}{2}\left(1+\eta_k^2-\eta_i^2-\eta_j^2\right)\,,\quad i\neq j\neq k\neq i\,,\\ \phi_{ij}=\delta_i+\delta_j+\alpha_{ij}+\mathrm{mod}(\pi),\quad i,j,k=1,2,3\,,\\ \sin\alpha_{ij}=\sqrt{\frac{1}{4\eta_i\eta_j}\left[\frac{\gamma_{ik}^2\gamma_{jk}^2}{\gamma_{ij}^2}-(\eta_i-\eta_j)^2\right]}\equiv X_{ij}\end{cases}$$

  - ☞ Solutions for α_{ij} can be either $\arcsin(X_{ij})$ or $\pi-\arcsin(X_{ij})$.
  - ☞ Boundary conditions on inelasticities

$$0\leq\eta_i\leq 1\,,\quad |1-\eta_j-\eta_k|\leq\eta_i\leq 1-|\eta_j-\eta_k|,\quad i\neq j\neq k\,.$$

- Asymptotic conditions on phase shifts and inelasticities:

$$\delta_i(s)=\delta_i(\infty)+[\delta_i(s_m)-\delta_i(\infty)]\frac{2}{1+(s/s_m)^{3/2}},\quad \eta(s)=\eta(\infty)+[\eta(s_m)-\eta(\infty)]\frac{2}{1+(s/s_m)^{3/2}}$$

Determinant of the MO matrix

❑ Unitarity relation

$$\begin{aligned}
&\mathrm{Im}\,\Omega(s+i\epsilon) = T^*(s+i\epsilon)\Sigma(s)\Omega(s+i\epsilon)\\
\Rightarrow\quad &\frac{\Omega(s+i\epsilon)-\Omega(s-i\epsilon)}{2i} = T^*(s+i\epsilon)\Sigma(s)\Omega(s+i\epsilon)\\
\Rightarrow\quad &\Omega(s+i\epsilon) = H(s+i\epsilon)\Omega(s-i\epsilon),\quad H(s)=\mathbb{1}+2iT(s+i\epsilon)\Sigma(s),\quad H(s)H^*(s)=\mathbb{1}.
\end{aligned}$$

☞ Phase space factors:

$$\Sigma = \mathrm{diag}\{\sigma_{ab}(s)\},\quad \sigma_{ab} = \frac{\lambda^{1/2}(s,m_a^2,m_b^2)}{s}\Theta[s-(m_a+m_b)^2]$$

☞ The determinant satisfies a single-channel Omnès-type relation

$$\det[\Omega(s+i\epsilon)] = e^{2i\phi(s)}\det[\Omega(s-i\epsilon)],\quad \exp 2i\phi(s) = \det[H(s)]$$

☞ Omnès matrix has unique solution, ensured by the asymptotic conditions [Moussallam, EPJC14, 2000]

$$\lim_{s\to\infty}|T_{ij}(s)| = 0 \quad\text{for}\quad i\neq j,\quad \lim_{s\to\infty}\sum_i^n(\delta_i(s)) = n\pi \Longrightarrow \lim_{s\to\infty}\det[\Omega(s)]\to 1/s^n$$

Fitting parameters in combined fit

- Summary of the fitting parameters: $\{\beta_0, \beta_1^B, \beta_2^B, \delta, \delta'\}$

$$\Omega^{(1,0)} \cdot \vec{\mathcal{P}}^{(1,0)}(s) \to \Omega^{(1,0)} \cdot \left\{ \frac{\beta_0\, \vec{\Gamma}}{s - s_p} + \vec{\mathcal{P}}^{(1,0)}(s) \right\}, \quad \vec{\Gamma} = (g_{DK}, g_{D_s\eta})^T$$

$$\frac{\beta_1^D}{\beta_1^B} = \sqrt{\frac{\bar{m}_D}{\bar{m}_B}}(1+\delta), \quad \frac{\beta_2^D}{\beta_2^B} = \sqrt{\frac{\bar{m}_B^3}{\bar{m}_D^3}}(1-3\delta), \quad f_P \to f_P \times (1+\delta')$$

Table: Properties of the $D_{s0}^*(2317)$ pole from the unitarized chiral amplitudes.

$\sqrt{s_p}$ [MeV]	g_{DK} [GeV]	$g_{D_s\eta}$ [GeV]
2315^{+18}_{-28}	$9.5^{+1.2}_{-1.1}$	$7.5^{+0.5}_{-0.5}$

Table: Results from the bottom-charm combined fit

	$\frac{\chi^2}{\text{dof}} = 2.77$	correlation matrix β_0	β_1^B	β_2^B	δ	δ'
β_0	0.152(14)(13)	1.000	0.502	0.499	−0.490	0.311
β_1^B	0.22(4)(4)	0.502	1.000	0.995	−0.965	0.848
β_2^B	0.0346(16)(15)	0.499	0.995	1.000	−0.958	0.845
δ	0.138(21)(18)	−0.490	−0.965	−0.958	1.000	−0.942
δ'	−0.18(4)(2)	0.311	0.848	0.845	−0.942	1.000

Bottom form factors and quadratic MO polynomials

- The scalar $\bar{B}\to\pi,\eta$ and $\bar{B}_s\to K$ form factors
  - Rank-two MO polynomials

$$\vec{\mathcal{P}}(s)=\vec{\alpha}_0+\vec{\alpha}_1 s+\vec{\alpha}_2 s^2\ .$$

  - Best fit resluts

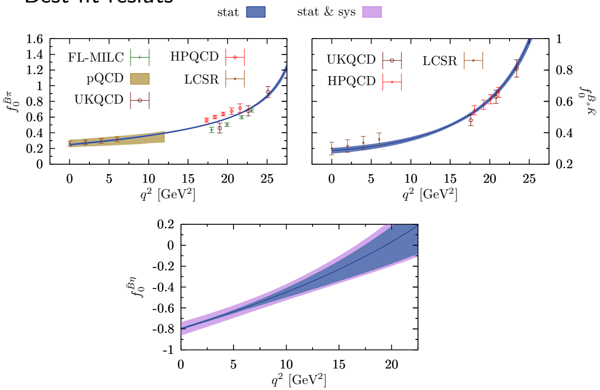


Table: parameters

χ^2/dof	3.7
β_1^B	0.74(22)
$(\beta_2\times 10)$	0.53(8)
$(\alpha_{2,1}\times 10^3)$	0.24(6)
$(\alpha_{2,2}\times 10^3)$	−0.1(7)
$(\alpha_{2,3}\times 10^3)$	1.0(8)

Differential decay rate

- For the semi-leptonic decay of $\bar{B}(p) \to D(p')\ell(q_1)\bar{\nu}_\ell(q_2)$:

$$
\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2} &= \frac{8\mathcal{N}|\vec{p}^{\,*}|}{3}\left[\left(1+\frac{m_\ell^2}{2q^2}\right)|H_0|^2+\frac{3m_\ell^2}{2q^2}|H_t|^2\right] \\
\mathcal{N} &= \frac{G_F^2}{256\pi^3}\eta_{\mathrm{EW}}^2|V_{cb}|^2\frac{q^2}{m_B^2}\left(1-\frac{m_\ell^2}{q^2}\right)^2
\end{aligned}
$$

  - Momenta: $q \equiv p - p'$ and $|\vec{p}^{\,*}|$ the modulus of the three-momentum of the D meson in rest frame of $\bar{B}$
  - Fermi coupling constant $G_F = 1.166 \times 10^{-5}$ GeV^{-2}
  - The factor $\eta_{\mathrm{EW}} = 1.0066$ accounts for the leading order electroweak corrections

- Helicity amplitudes

$$
H_0 = \frac{2m_B|\vec{p}^{\,*}|}{\sqrt{q^2}}f_+(q^2)\ , \quad H_t = \frac{m_B^2-m_D^2}{\sqrt{q^2}}f_0(q^2)\ .
$$

PKU representation

[Xiao, Zheng, Zhou, et al, NPA (2001), NPA(2004), JHEP(2005)]

- A virtual/bound state pole at s_0 (real), $s_L < s_0 < s_R$:
  - Scattering length:

$$a(s_0) = \pm \frac{2\sqrt{s_R}}{s_R - s_L}\sqrt{\frac{s_0 - s_L}{s_R - s_0}} \ .$$

  - S matrix and phase shift:

$$S(s) = \frac{1 \pm i\rho(s)a(s_0)}{1 \mp i\rho(s)a(s_0)} \ , \quad \delta(s) = \mp \arctan\left[\frac{s\rho(s)}{s - s_L}\sqrt{\frac{s_0 - s_L}{s_R - s_0}}\right]$$

- A resonance located at z_0 and z_0^*.
  - The S matrix and phase shift

$$S(s) = \frac{M^2(z_0) - s + i\rho(s)sG[z_0]}{M^2(z_0) - s - i\rho(s)sG[z_0]} \ , \quad \delta(s) = \arctan\left[\frac{s\rho(s)G}{M^2 - s}\right]$$

$$M^2(z_0) = \mathrm{Re}[z_0] + \frac{\mathrm{Im}[z_0]\mathrm{Im}[z_0\rho(z_0)]}{\mathrm{Re}[z_0\rho(z_0)]} \ , \quad G[z_0] = \frac{\mathrm{Im}[z_0]}{\mathrm{Re}[z_0\rho(z_0)]} \ .$$