Enabling lattice calculation of TMDs via factorization

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Support:

Based on:
2004.14831
2201.08401
2205.12369

Motivation

SLAC ➤ DESY ➤ **CERN** ➤ **JLab** ➤ **EIC** ➤ …

1960s → **2030s**

1D PDFs ➤ **3D TMDs, GPDs** ➤ 5D GTMDs? ➤ …

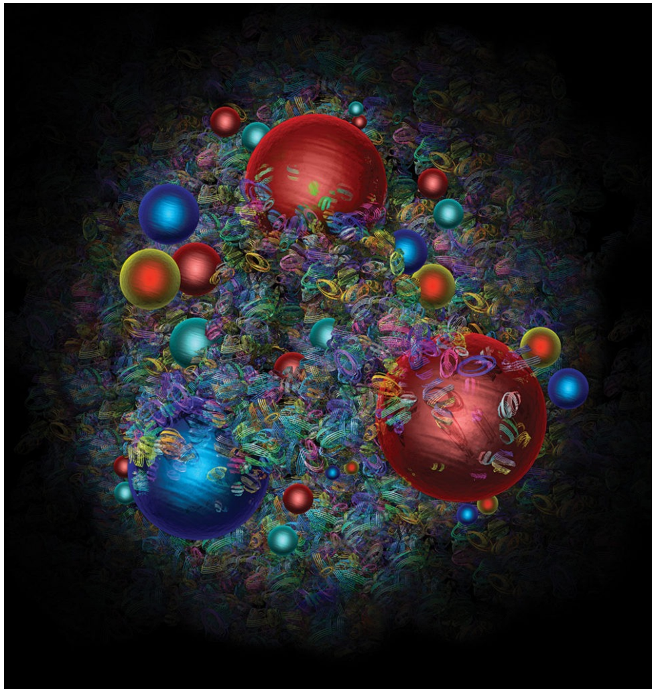


Figure: CERN

Soon, we'll have even higher precision experimental data about the proton's full 3D internal structure…

It's crucial to develop a corresponding first-principles understanding!

Roadmap for today's lecture

1. Background
2. Factorization
3. Implications

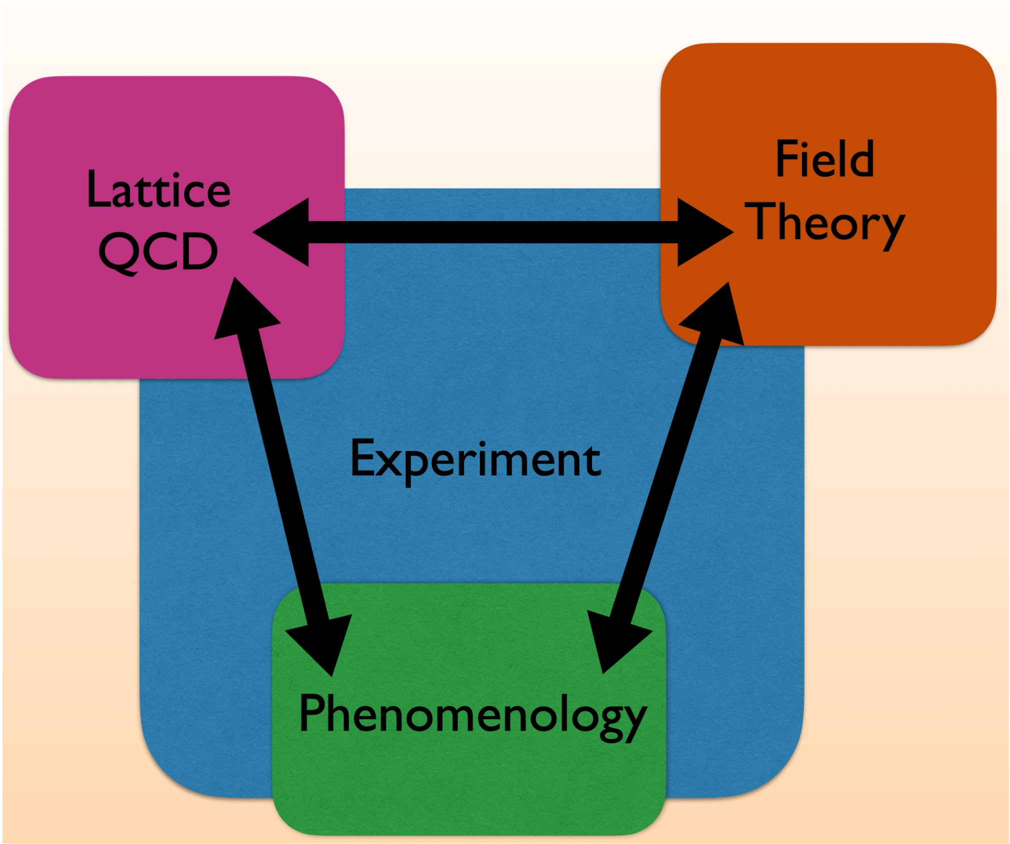
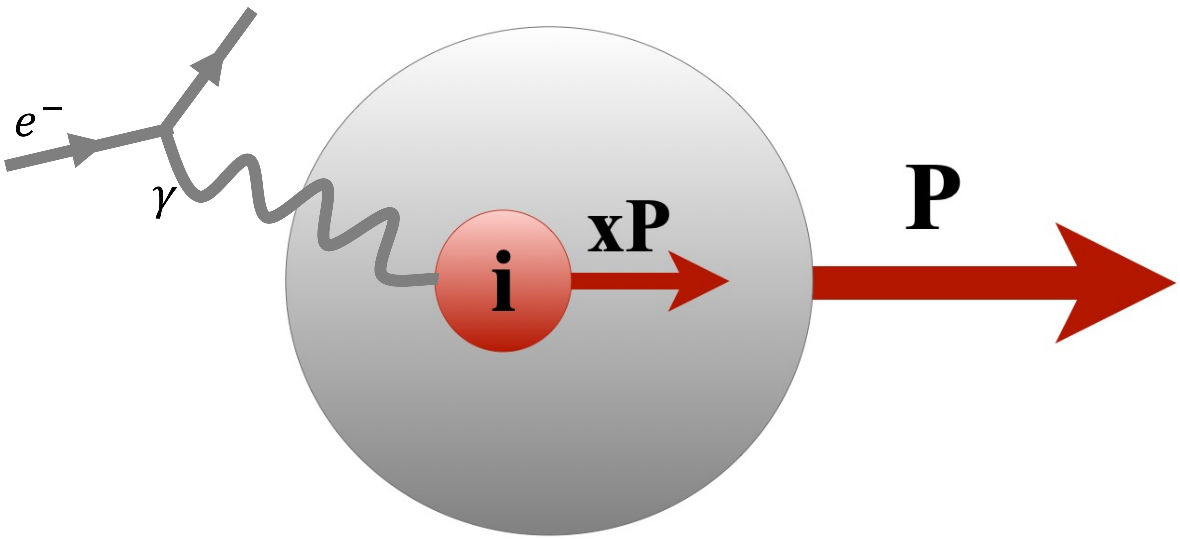
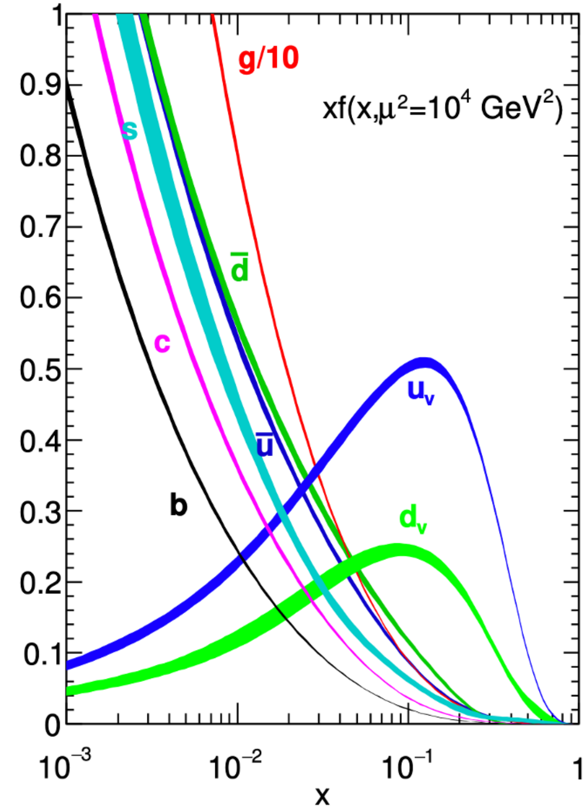


Figure: I. Stewart

PDFs

Parton Distribution Functions: 1D momentum distribution of quarks and gluons inside the proton



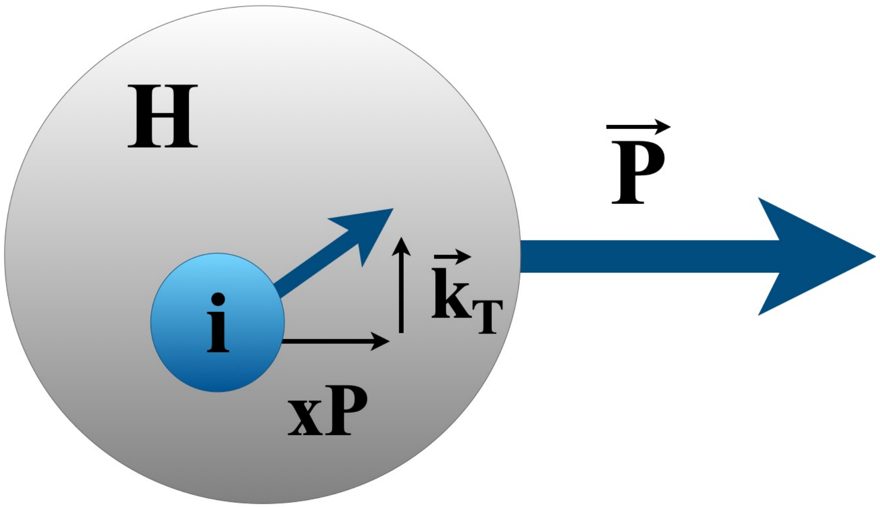


- Universal factors in processes
- SLAC-MIT experiment (1969): deep inelastic scattering

Figure: PDG

TMDs

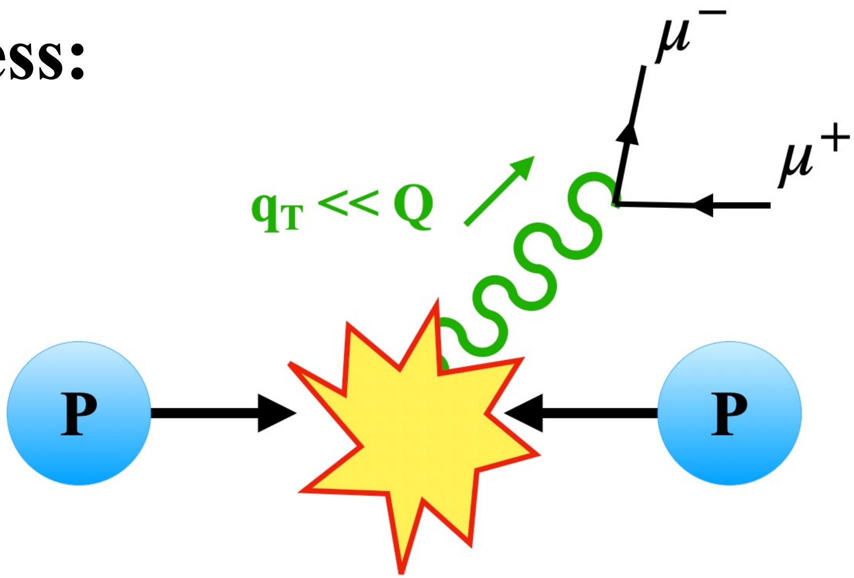
Transverse momentum dependent PDFs: full 3D picture



- Key factor in SIDIS, Drell-Yan, W/Z production, Higgs, …
- Challenge: important non-perturbative contributions ***even at*** perturbative scales

TMDs in cross-sections

Drell-Yan process:



Tree-level — Virtual corrections

$$\frac{d\sigma^{DY}}{dQ^2 dY\, d^2\vec{q}_T} = \sigma_0 \sum_{i,j} H_{ij}(Q,\mu)$$

TMDs

$$\times \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} f_{i/h_1}(x_1,\vec{b}_T,\mu,\zeta_1)\, f_{j/h_2}(x_2,\vec{b}_T,\mu,\zeta_2)$$

Collins-Soper (CS) scale:

$\zeta = 2(xP^+ e^{-y_n})^2$

Collins, *Foundations of Perturbative QCD*.

Evolution of TMD scales

Possible to relate TMDs at different scales $(\boldsymbol{\mu}, \boldsymbol{\zeta})$ & $(\boldsymbol{\mu_0}, \boldsymbol{\zeta_0})$:

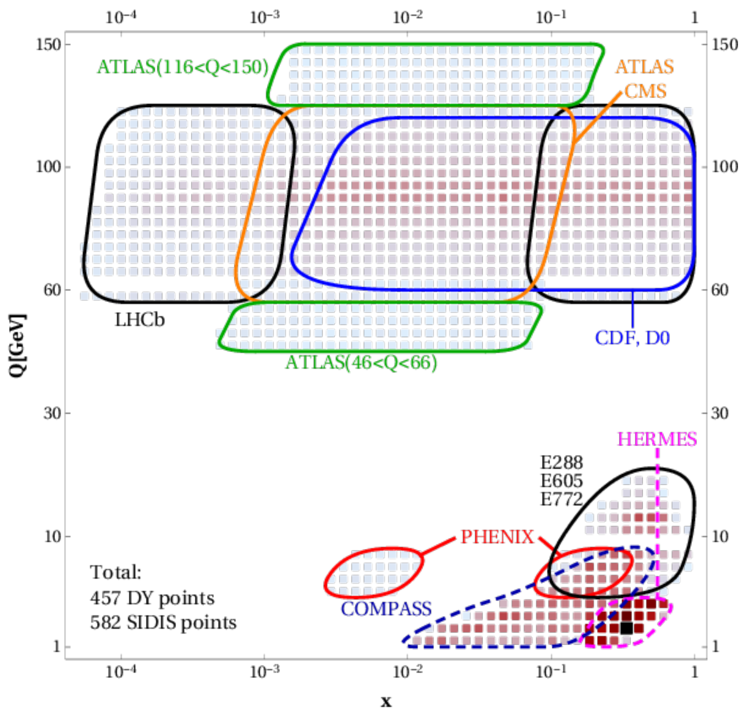
$$\boldsymbol{f_q}(\boldsymbol{x}, \vec{\boldsymbol{b}}_{\boldsymbol{T}}, \boldsymbol{\mu}, \boldsymbol{\zeta}) = \exp\left[\int_{\boldsymbol{\mu_0}}^{\boldsymbol{\mu}} \frac{d\mu'}{\mu'} \gamma_\mu^q(\mu', \zeta_0)\right] \exp\left[\frac{1}{2} \underbrace{\boldsymbol{\gamma_\zeta^q}}_{\text{CS kernel}}(\mu, b_T) \ln \frac{\boldsymbol{\zeta}}{\boldsymbol{\zeta_0}}\right] \boldsymbol{f_q}(\boldsymbol{x}, \vec{\boldsymbol{b}}_{\boldsymbol{T}}, \boldsymbol{\mu_0}, \boldsymbol{\zeta_0})$$

Experiment	$\mu, \sqrt{\zeta} \sim Q$
Lattice	$\mu, \sqrt{\zeta} \sim 1\ \mathrm{GeV}$

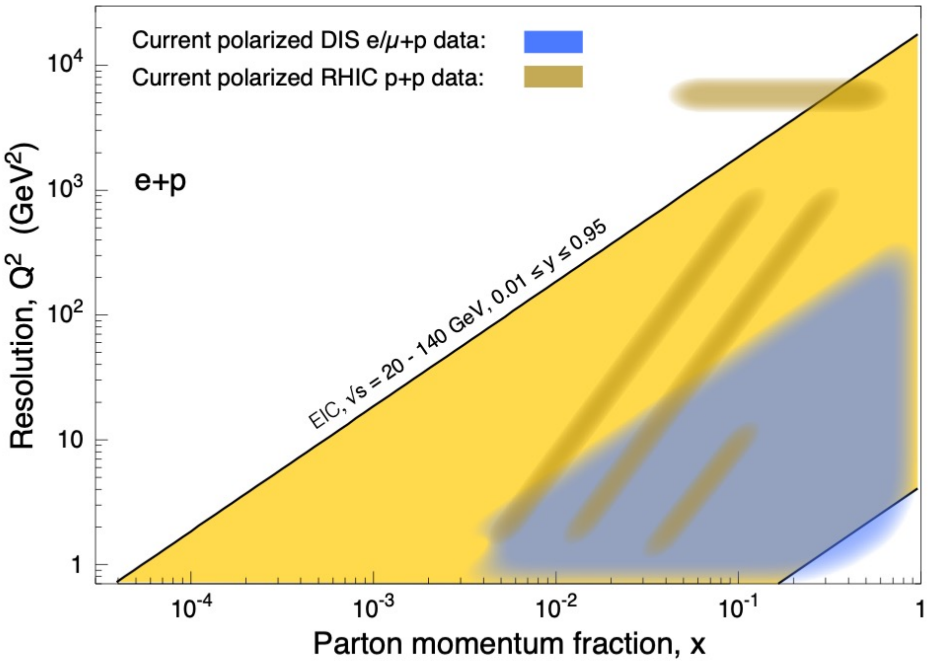
Collins & Soper (1982, 1983). Collins, Soper, & Sterman (1985).

TMDs from experiment

Data used in Scimemi & Vladimirov global fit



Projected EIC data



Figures: Scimemi & Vladimirov (JHEP 2019). EIC Yellow Report (2021).

Global fits to experiment

Split TMD into two pieces:

$$f_i(x, b_T, \mu, \zeta) = \boldsymbol{f_i^P[x, b^*(b_T), \mu, \zeta]} \ \boldsymbol{f^{NP}(x, b_T, \zeta)}$$

Perturbative piece:

- Expand in $\alpha_s(b_T^{-1})$ about collinear PDF
- Known to three loops! Ebert, Mistlberger, Vita (JHEP 2020). Luo, Yang, Zhu, Zhu (JHEP 2021).

Non-perturbative piece:

- Construct model, fit to data e.g., JAM collaboration

Example non-perturbative model

Example: 11-parameter model

TMD-PDF

$$f_{NP}(x,b) = \exp\left(-\frac{\lambda_1(1-x)+\lambda_2 x + x(1-x)\lambda_5}{\sqrt{1+\lambda_3 x^{\lambda_4} b^2}}\, b^2\right)$$

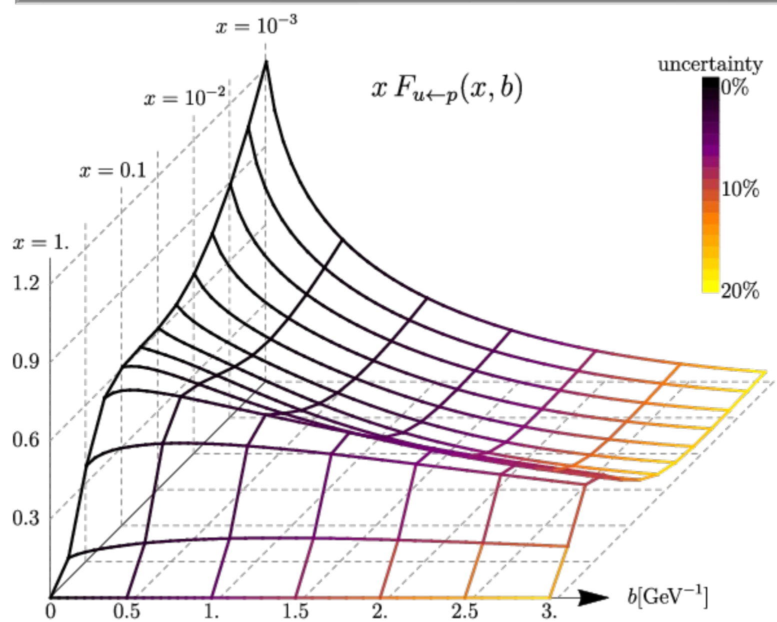
TMD-FF

$$D_{NP}(x,b) = \exp\left(-\frac{\eta_1 z+\eta_2(1-z)}{\sqrt{1+\eta_3 (b/z)^2}}\frac{b^2}{z^2}\right)\left(1+\frac{\eta_4 b^2}{z^2}\right)$$

CS kernel

$$\gamma_\zeta^q(\mu,b) = \gamma_\zeta^{q\,pert}(\mu,b^*) - \frac{1}{2}c_0 b b^*$$

$$b^*(b) = \frac{b}{\sqrt{1+b^2/B_{NP}^2}}$$



Scimemi & Vladimirov (JHEP 2019).

- Describes data well at wide range of energy scales
- ***But*** large uncertainties from experiment at high b
- Uncertainties from choice of functional form are ***not*** known

Example: global fits of the CS kernel

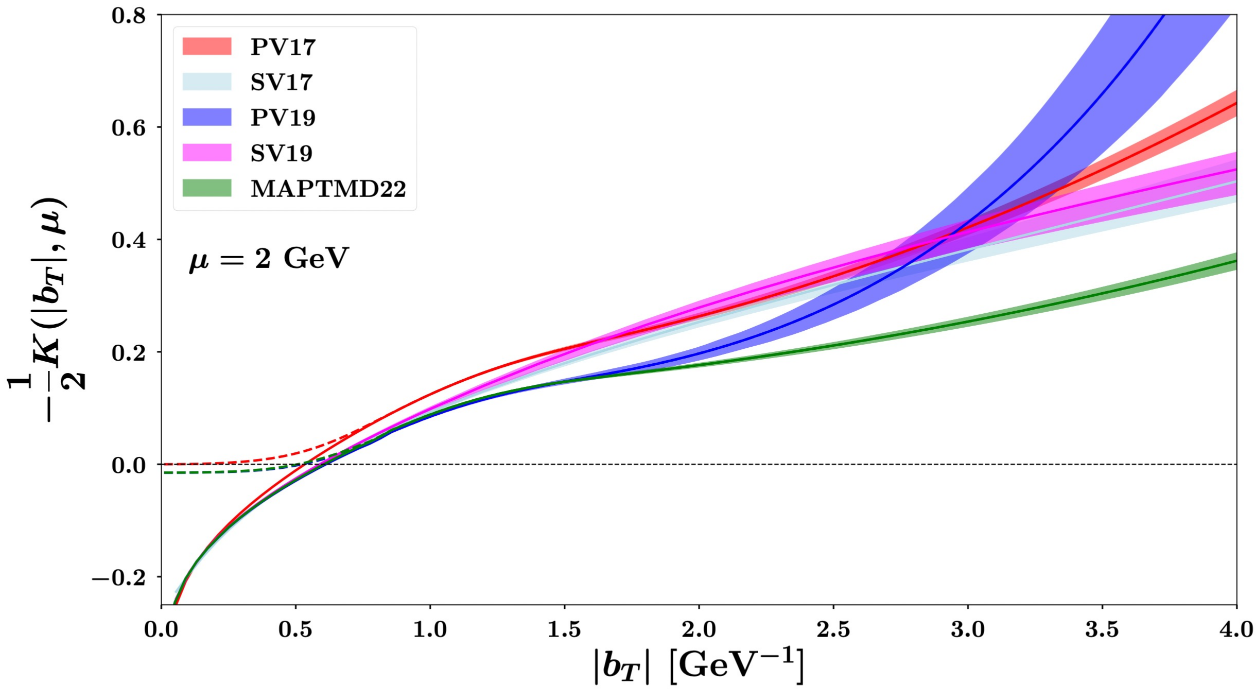


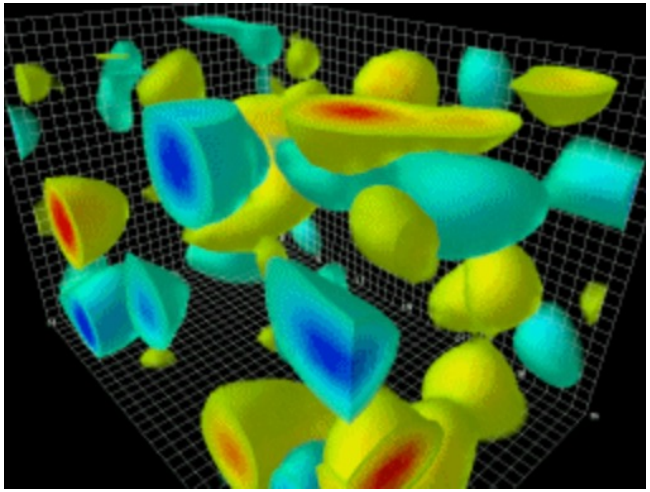
Figure: MAP collaboration, Bacchetta et al. (2022).

- Large non-perturbative contributions to TMDs
- At low b_T, good fits; agree by construction
- Larger uncertainty in non-perturbative region

Lattice QCD in a nutshell

General premise:

- ➢ Discretize QFT to regulate divergences
- ➢ Only known systematically improvable numerical approach



Figures: Argonne, D. Leinweber (QCD vacuum).

Want correlation functions:

$$\langle \mathcal{O} \rangle = \frac{\int [dU]\, e^{-S[U]} \mathcal{O}}{\int [dU]\, e^{-S[U]}}$$

Generate representative set of gauge configurations using Monte Carlo

Recipe for TMDs on the lattice

TMD in cross-sections

Project to equal-time slice & discretize

Lattice-calculable TMD

Run simulations

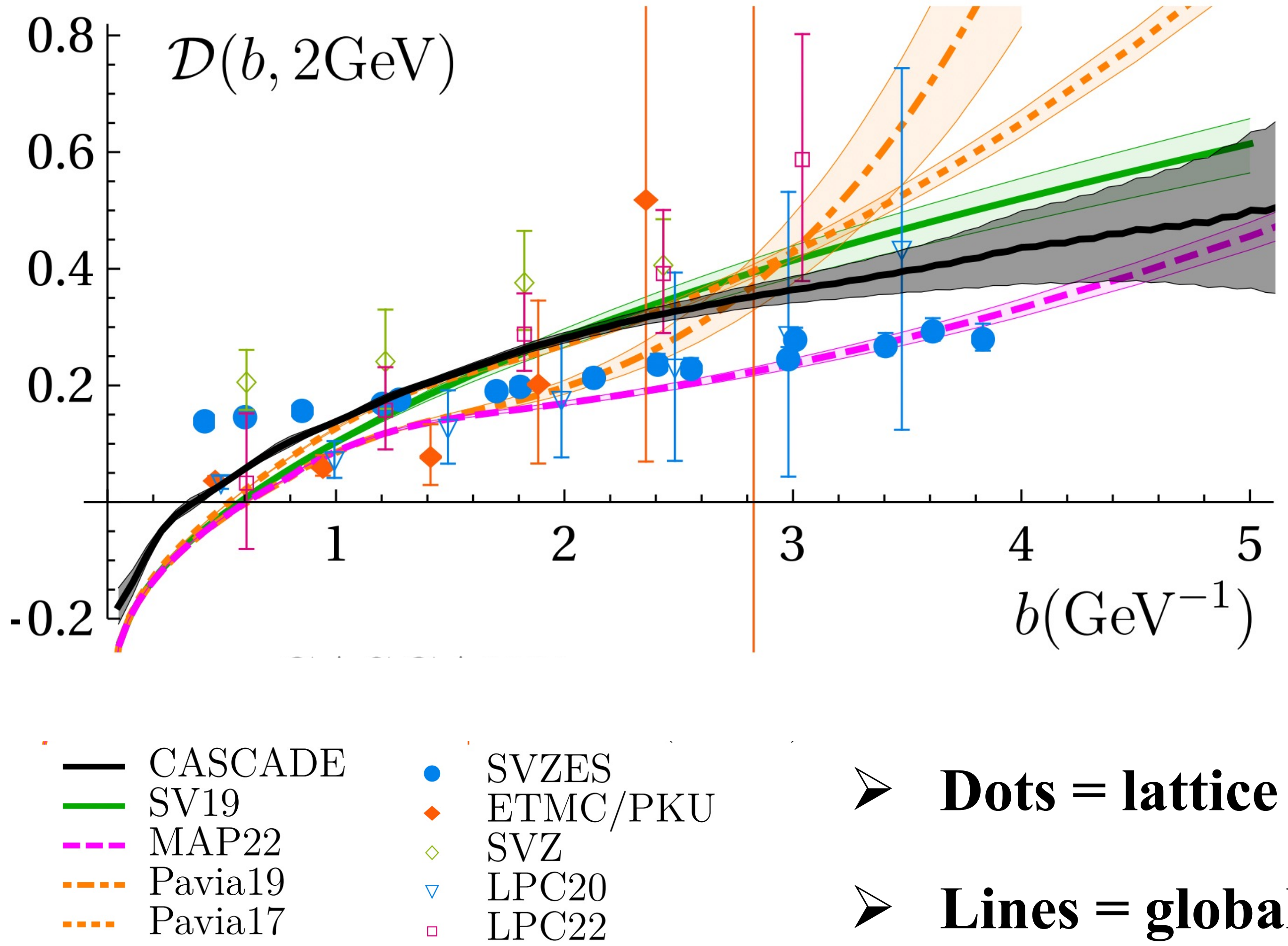
Raw lattice TMD data

Limit of zero lattice spacing, ∞ size

Continuum extrapolation of lattice TMD

Factorization formula

CS kernel from the lattice



- **Dots = lattice data**
- **Lines = global fits**

Martinez & Vladimirov (2022).

Why lattice QCD?

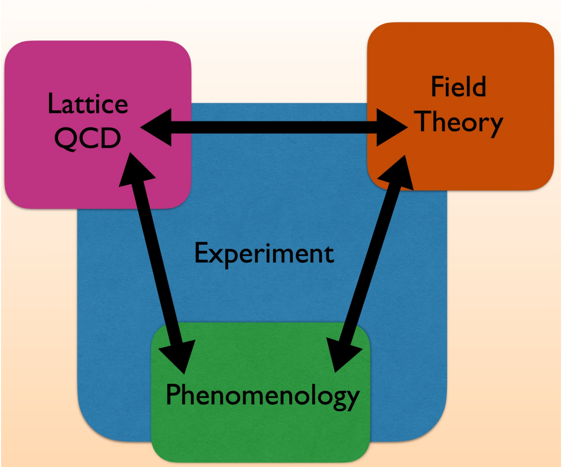
Complementary to experiment & phenomenology:

- Good to check that QFT and experiment match

- Easier to access CS kernel, spin and flavor dependence than in experiment

- Can improve global fit errors with lattice data

- Calculations beyond $b_T > 1\ \mathrm{GeV}^{-1}$



TMDs from field theory

Many schemes to define TMDs…

Modern Collins

$$\tilde{f}_{i/p}(x,\mathbf{b}_T,\mu,\zeta) = \lim_{\epsilon\to 0} Z_{\mathrm{uv}}(\mu,\zeta,\epsilon) \lim_{y_B\to-\infty} \frac{\tilde{f}^{0\,(\mathrm{u})}_{i/p}(x,\mathbf{b}_T,\epsilon,y_B,xP^+)}{\sqrt{\tilde{S}^{0}\cdots}}$$

Echevarria, Idilbi, Scimemi

$$\tilde{f}_{i/p}(x,\mathbf{b}_T,\mu,\zeta) = \lim_{\substack{\epsilon\to 0\\ \delta^+\to 0}} Z^i_{\mathrm{uv}}(\mu,\zeta,\epsilon) \frac{\tilde{f}^{0\,(\mathrm{u})}_{i/p}(x,\mathbf{b}_T,\epsilon,\delta^+/(xP^+))}{\sqrt{\tilde{S}^0_{\mathrm{EIS}}(b_T,\epsilon,\delta^+ e^{-y_n})}}$$

Chiu, Jain, Neill, Rothstein

$$\tilde{f}_{i/p}(x,\mathbf{b}_T,\mu,\zeta) = \lim_{\substack{\epsilon\to 0\\ \eta\to 0}} Z^i_{\mathrm{uv}}(\mu,\zeta,\epsilon)\, \tilde{f}^{0\,(\mathrm{u})}_{i/p}(x,\mathbf{b}_T,\epsilon,\eta,xP^+) \sqrt{\tilde{S}^0_{\mathrm{CJNR}}(b_T,\epsilon,\eta)}$$

Becher & Neubert

$$\lim_{\substack{\epsilon\to 0\\ \alpha\to 0}} \left[\tilde{f}^{0(\mathrm{u}),\mathrm{BN}}_{i/p}(x_1,\mathbf{b}_T,\epsilon,\alpha,x_a P_A^+)\tilde{f}^{0(\mathrm{u}),\mathrm{BN}}_{j/p}(x_2,\mathbf{b}_T,\epsilon,\alpha,x_b P_B^-)\right]$$

$$\left(b_T^2 Q^2\right)^{-\gamma^q_\zeta(\mu,b_T)} \left[\cdots {}_T,\mu,\zeta = b_0^2/b_T^2)\right]$$

Ji, Ma, Yuan

$$\tilde{f}_{i/p}(x_a,\mathbf{b}_T,\mu,x_a\tilde{\zeta}_a;\rho) = \lim_{\epsilon\to 0} Z^i_{\mathrm{uv}}(\mu,\rho,\epsilon) \frac{\tilde{f}^{0\,(\mathrm{u})}_{i/p}(x_a,\mathbf{b}_T,\epsilon,v,xP^+)}{\sqrt{\tilde{S}^0_{v\bar{v}}(b_T,\epsilon,\rho)}} + O(v^+,\bar{v}^-)\,.$$

Etc!

First goal: sort this out.

Definition of TMDs in QFT

TMD

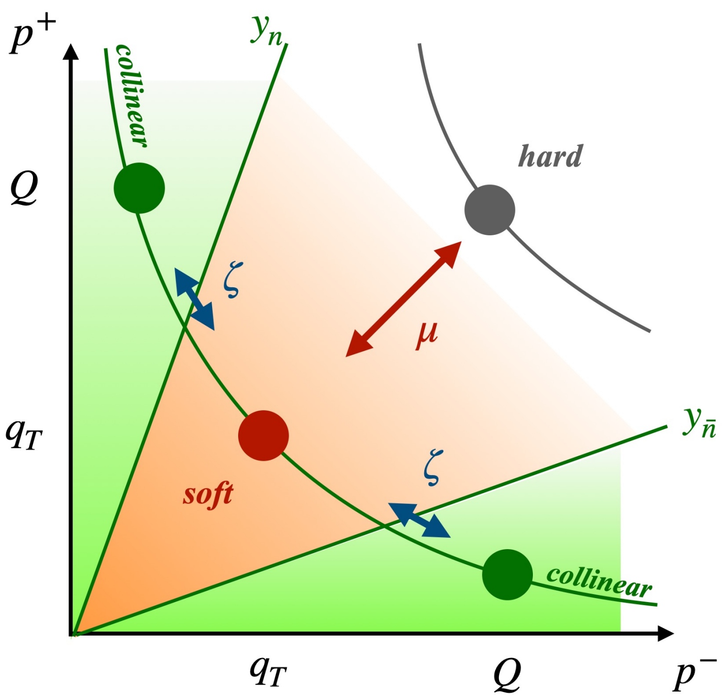
$$f = \lim \; Z_{UV} \frac{B^{[\Gamma]}_{q_i/H}}{\sqrt{S^R}}$$

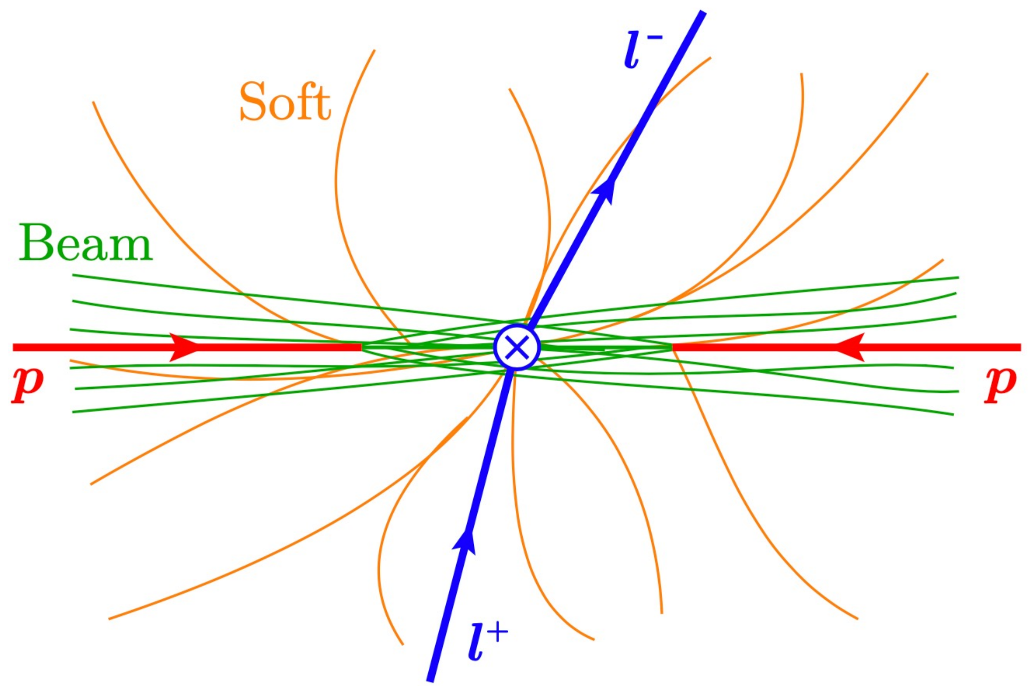
lightcone, renormalization

Scheme-dependent renormalization

Beam function

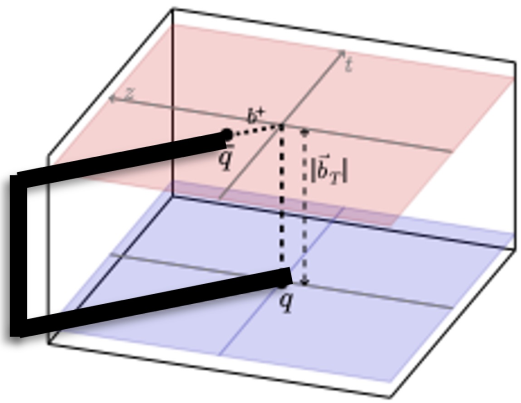
Soft factor



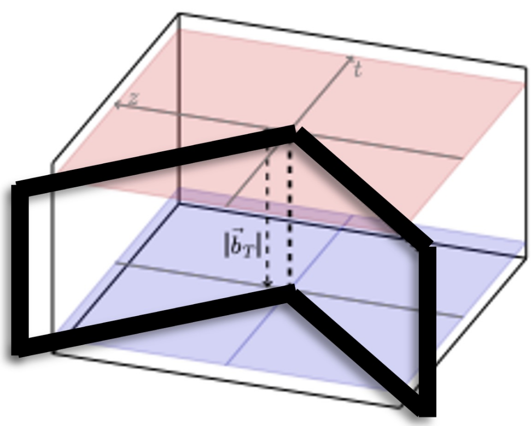


TMD matrix elements

Beam function:

$\langle P|$  $|P\rangle$

Soft factor:

$\langle 0|$  $|0\rangle$

- **Analog of parton in QCD:** quark field attached to lightcone Wilson line
- **Soft & collinear particle interactions:** approximated by gauge links
- **Gauge invariance:** need closed paths

New: unified notation

Can describe lattice & continuum off-lightcone schemes using the same generic **beam function** & **soft factor**

- Quasi-TMD
- Collins TMD
- JMY scheme
- Etc.

$$f = \lim_{\substack{\text{lightcone,}\\ \text{renormalization}}} Z_{UV} \frac{B^{[\Gamma]}_{q_i/H}(b, P, \epsilon, \eta v, \delta)}{\sqrt{S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})}}$$

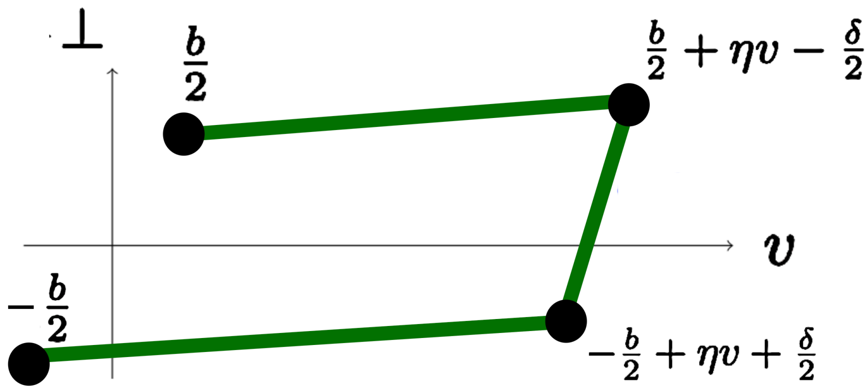
Each scheme is characterized by a distinct set of **arguments** & **limits**

Ebert, **Schindler**, Stewart, and Zhao (2022).

Meaning of the correlators

$$\textbf{Beam} = \left\langle P \left| \bar{q}_i \frac{\Gamma}{2} W_{\sqsupset}^{F}(b, \eta v, \delta) q_i \right| P \right\rangle$$

$$\textbf{Soft} = \frac{1}{d_R} \langle 0 | \mathrm{Tr}[S_{\gtrless}^{R}(b, \eta v, \overline{\eta v})] | 0 \rangle$$



- **b^μ, ηv^μ, δ^μ:** parametrize Wilson lines
- **Length η:** finite (lattice) or infinite (physical TMD)
- $\delta^\mu = (0,0,0,\tilde{b}^z)$ for quasi
  $= (0,0,0,0)$ for MHENS

Ebert, **Schindler**, Stewart, and Zhao (2022).

Now, neat & tidy tables of schemes

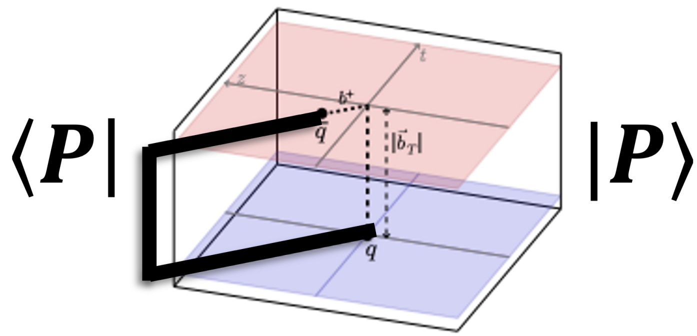
	Collins scheme	Quasi-TMDs
TMD	$\lim\limits_{\epsilon\to 0} Z_{\mathrm{UV}}^{\kappa_i} \lim\limits_{y_B\to-\infty} \frac{B_{i/h}}{\sqrt{S^{\kappa_i}}}$	$\lim\limits_{a\to 0} Z_{\mathrm{UV}}^{\kappa_i} \frac{\tilde{B}_{i/h}}{\sqrt{S^{\kappa_i}}}$
Beam	$\Omega_{i/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right] \xrightarrow{\mathrm{FT}} B_{i/h}$	$\Omega_{i/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z}) \xrightarrow{\mathrm{FT}} \tilde{B}_{i/h}$
Soft	$S^{\kappa_i}\left[b_\perp, \epsilon, -\infty n_A(2y_n), -\infty n_B(2y_B)\right]$	$S^{\kappa_i}\left[b_\perp, a, -\tilde{\eta}\frac{n_A(2y_n)}{\lvert n_A(2y_n)\rvert}, -\tilde{\eta}\frac{n_B(2y_B)}{\lvert n_B(2y_B)\rvert}\right]$
b^μ	$(0, b^-, b_\perp)$	$(0, b_T^x, b_T^y, \tilde{b}^z)$
v^μ	$(-e^{2y_B}, 1, 0_\perp)$	$(0, 0, 0, -1)$
δ^μ	$(0, b^-, 0_\perp)$	$(0, 0, 0, \tilde{b}^z)$
P^μ	$\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{-y_P}, 0_\perp)$	$m_h(\cosh y_{\tilde{P}}, 0, 0, \sinh y_{\tilde{P}})$

Ebert, **Schindler**, Stewart, and Zhao (2022). **Schindler**, Stewart, and Zhao (2022).

TMDs from the lattice?

Naïve lattice QCD:

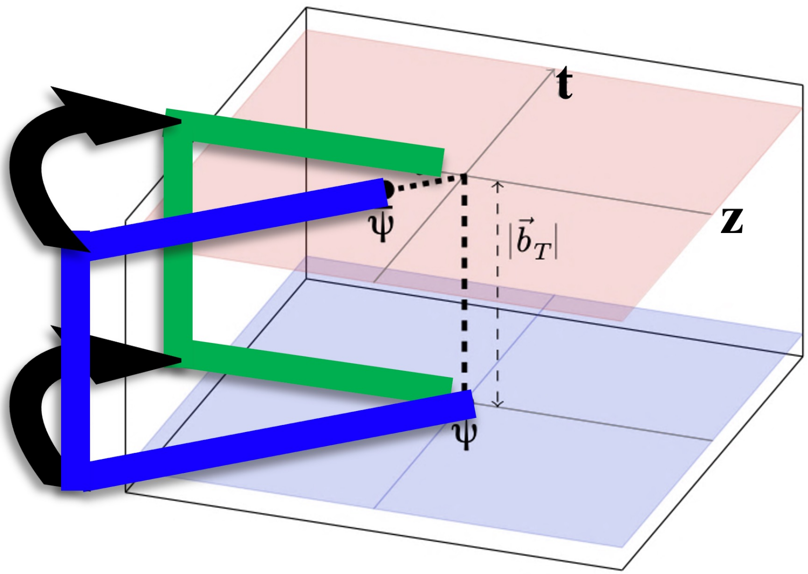
1. Make TMD Wilson lines finite
2. Rotate to Euclidean space
3. Discretize path integral
4. Run Monte Carlo simulations
5. Extrapolate results back to continuum



Problem: Wilson lines are on the lightcone

Real Minkowski time variable → complex Euclidean action

Circumventing the sign problem



Trick: Project the desired **physical** Wilson line onto an **equal-time slice**

(Nontrivial! More later.)

- **Lattice TMD** is numerically tractable
- Want **physical TMD** & **"lattice TMD"** to be same in IR
- At worst, differ in UV & related by perturbative matching

Things are progressing rapidly…

2013	**First lattice TMD (MHENS scheme) proposed** Musch, Hägler, Engelhardt, Negele, and Schäfer
2014	**New lattice TMD (quasi-TMD) proposed, 1-loop studies** Xiangdong Ji
⋮	**Lattice calculations of MHENS beam functions** MHENS and collaborators
2018	**Quasi-TMD theory put on firmer footing** Ebert, Stewart, and Zhao
2019	**Proposal for lattice calculation of quasi-soft function** Ji, Liu, and Liu
⋮	**First lattice results for CS kernel & quasi-soft function** MIT, LPC, ETMC, and Regensburg lattice groups
2022	**Factorization connecting quasi & physical TMDs** Ebert, **Schindler**, Stewart, and Zhao

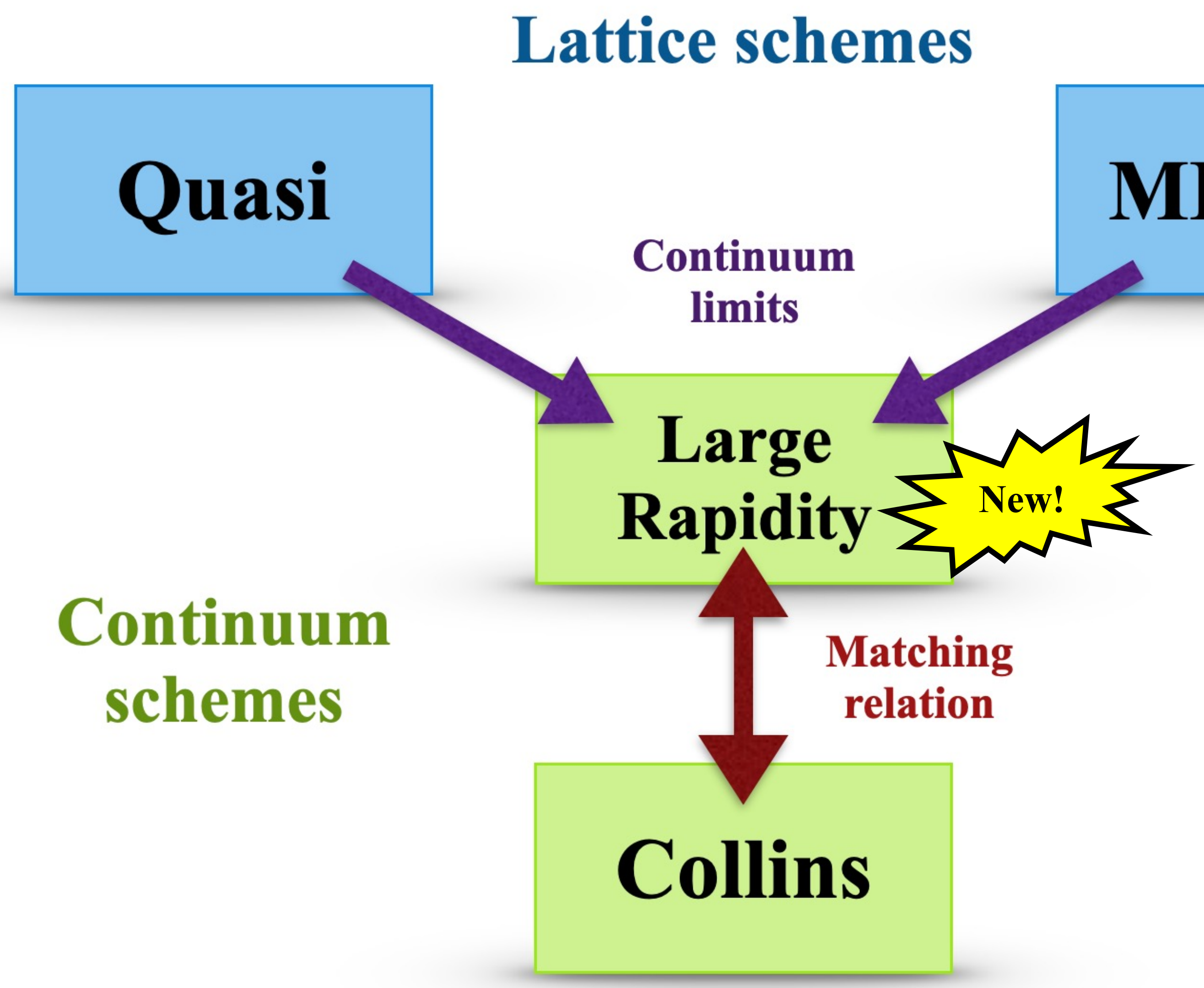
TMD factorization

Drell-Yan cross-section	$d\sigma = H \int f \otimes f$	$q_T \ll Q$
Formal definition	$f = Z_{UV} \dfrac{B}{\sqrt{S}}$	SCET/QCD $q_T \ll Q$
On the lattice	Goal $f = C \times \tilde{f}_{lattice}$	LaMET $P^z \gg \Lambda_{QCD}$

Collins, *Foundations of Perturbative QCD*. LaMET: Ji (2014).

Proof of factorization

Connecting physical & lattice TMDs

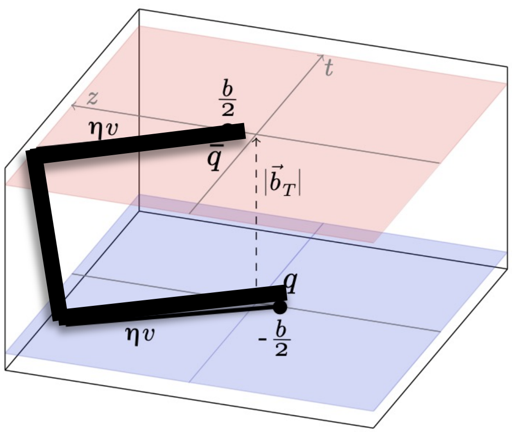


Ebert, **Schindler**, Stewart, and Zhao (2022).

Two main lattice approaches

Today

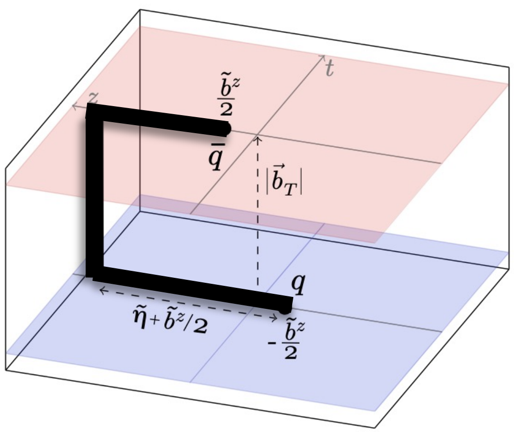
MHENS scheme



- Pioneered lattice TMDs
- Focused on *x*-moments
- Renormalization, soft function, factorization not fully known

Musch, Hägler, Engelhardt, Negele, Schäfer (2013).

Quasi-TMDs



- Newer; fewer results for proton
- Focused on full TMD
- Renormalization, soft function have been proposed

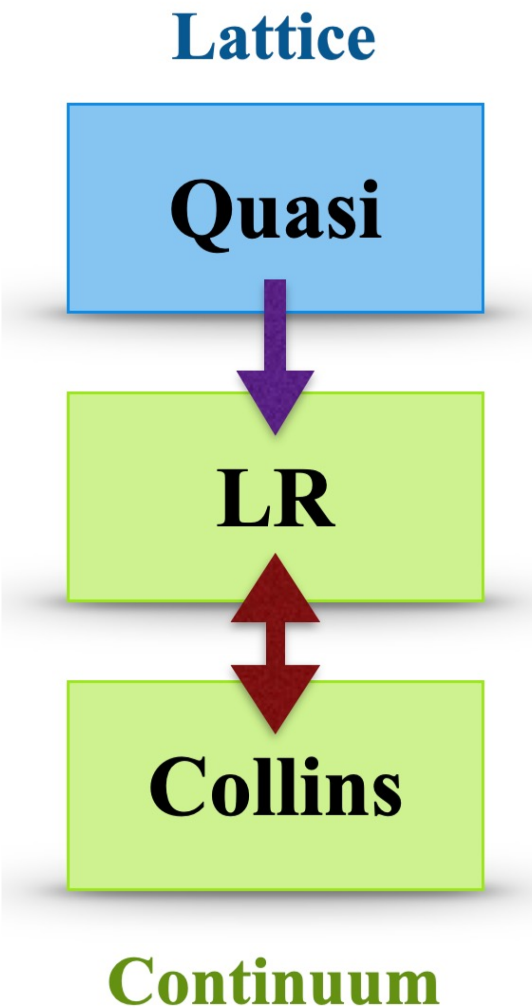
Ji (2014).

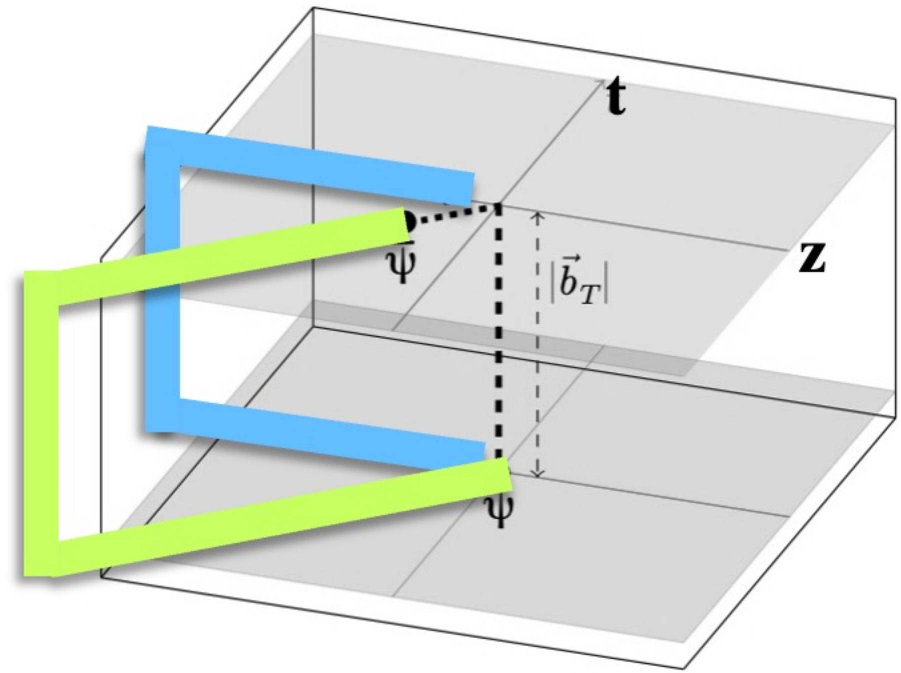
Physical TMD schemes in this talk

	Collins	Large Rapidity (LR)
Limits	$\lim_{\epsilon\to 0} \lim_{y_B\to -\infty} Z_{UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\lim_{-y_B\gg 1} \lim_{\epsilon\to 0} Z_{UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$
Beam	$\boldsymbol{\Omega}_{\boldsymbol{q/h}}^{[\gamma^+]}[b, P, \epsilon, -\infty n_B(y_B), b^- n_b]$	$\boldsymbol{\Omega}_{\boldsymbol{q/h}}^{[\gamma^+]}[b, P, \epsilon, -\infty n_B(y_B), b^- n_b]$
Soft	$\boldsymbol{S}^{\boldsymbol{R}}[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$	$\boldsymbol{S}^{\boldsymbol{R}}[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$

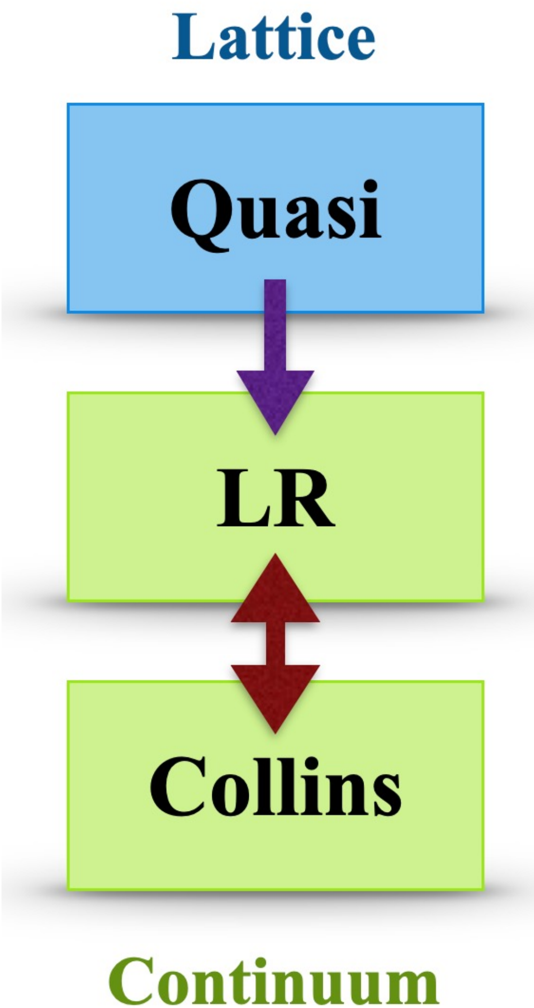
- Closely related to lattice TMDs
- Regularize by taking off lightcone (characterize by rapidity y_B)
- Only differ by an order of limits

Definition of the schemes





Factorization derivation steps



Step 1: same at large rapidity $P^z >> \Lambda_{QCD}$

- Expand & relate their variables
- Take Wilson line length $|\eta| \to \infty$

Step 2: need a matching coefficient

- Different UV renormalizations
- Nontrivial relationship

Focus on beams: quasi-soft function is chosen to reproduce the Collins soft function

Step 1: Quasi to Large Rapidity

Compare Lorentz invariants formed from beam function arguments b^μ, P^μ, δ^μ, ηv^μ

Use boosts to show quasi = LR as $|\boldsymbol{\eta}| \to \infty$ **& $P^z >> \Lambda_{QCD}$**

	Quasi	**LR**
b^2	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2$
$(\eta v)^2$	$-\tilde{\eta}^2$	$-2\eta^2 e^{2y_B}$
$P \cdot b$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$	$\sinh(y_P - y_B) \operatorname{sgn}(\eta)$
$\frac{\delta^2}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1
P^2	m_h^2	m_h^2

Quasi to LR: same at Large Rapidity

Matching up Lorentz invariants implies:

$$\sinh(\tilde{y}_P)\mathrm{sgn}(\eta) = \sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$$

Need $y_P - y_B = y_{\tilde{P}}$

	Quasi	LR
b^2	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2$
[illegible]	$-\tilde{\eta}^2$	$-2\eta^2 e^{2y_B}$
[illegible]	$m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$
$\frac{b\cdot(\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}}\,\mathrm{sgn}(\eta)$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T}\,\mathrm{sgn}(\eta)$
$\frac{P\cdot(\eta v)}{\sqrt{P^2\lvert\eta v\rvert^2}}$	$\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$	$\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$
$\frac{[illegible]}{b^2}$	$\frac{[illegible]}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b\cdot\delta}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P\cdot\delta}{P\cdot b}$	1	1
$\frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}$	1	1
P^2	m_h^2	m_h^2

Quasi to LR: same at Large Rapidity

Previous slide:

$$y_B = y_{\tilde{P}} - y_P$$

Need:

$$-m_h \tilde{b}_z \sinh y_{\tilde{P}} = \frac{m_h}{\sqrt{2}} b^- e^{y_P}$$

Finite $P \cdot b$ & $y_P \to$ finite b^-

For quasi, $y_{\tilde{P}} \to \infty$, so $\tilde{b}^z \to 0$

	Quasi	LR
b^2	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2$
$(\eta v)^2$	$-\tilde{\eta}^2$	$-2\eta^2 e^{2y_B}$
$P \cdot b$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$
$\frac{b\cdot(\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$\frac{[illegible]}{\sqrt{(\tilde{b}^z)^2 + b_T^2}}\,\mathrm{sgn}(\eta)$	$-\frac{[illegible]}{\sqrt{2}\, b_T}\,\mathrm{sgn}(\eta)$
$\frac{P\cdot(\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$	$\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$
$\frac{\delta^2}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b\cdot\delta}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P\cdot\delta}{P\cdot b}$	1	1
$\frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}$	1	1
P^2	m_h^2	m_h^2

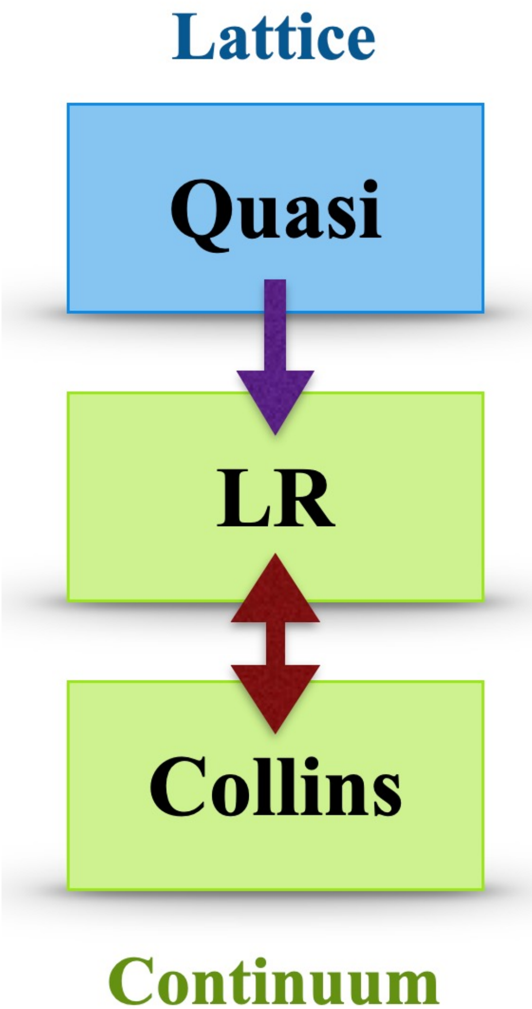
Quasi to LR: same at Large Rapidity

Need $\tilde{\eta} = \sqrt{2}\, e^{y_B} \eta$

In $y_{\tilde{P}} \to -\infty$ limit, $b_T \gg \tilde{b}_z$

	Quasi	**LR**
b^2	b^2 [illegible] $(\tilde{b}^z)^2$	b^2
$(\eta v)^2$	$-\tilde{\eta}^2$	$-2\eta^2 e^{2y_B}$
$P \cdot b$	$-m_h b^z \sinh y_{\tilde{P}}$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}}\,\mathrm{sgn}(\eta)$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T}\,\mathrm{sgn}(\eta)$
$\frac{P \cdot (\eta v)}{\text{[illegible]}}$	$\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$	$\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$
$\frac{\delta^2}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1
P^2	m_h^2	m_h^2

Factorization derivation steps



Step 1: Same at large rapidity

Step 2: need a matching coefficient

- Different UV renormalizations
- Nontrivial relationship

Step 2: Large Rapidity to Collins

	Collins	Large Rapidity (LR)
Limits	$\lim\limits_{\epsilon\to 0}\ \lim\limits_{y_B\to-\infty} Z_{UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\lim\limits_{-y_B\gg 1}\ \lim\limits_{\epsilon\to 0} Z_{UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$
Beam	$\boldsymbol{\Omega}_{\boldsymbol{q/h}}^{[\gamma^+]}[b, P, \epsilon, -\infty n_B(y_B), b^- n_b]$	$\boldsymbol{\Omega}_{\boldsymbol{q/h}}^{[\gamma^+]}[b, P, \epsilon, -\infty n_B(y_B), b^- n_b]$
Soft	$\boldsymbol{S^R}[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$	$\boldsymbol{S^R}[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$

- Order of UV limits **cannot** affect IR physics
- But if non-commuting → perturbative matching coefficient
- Non-commutativity can arise from divergences

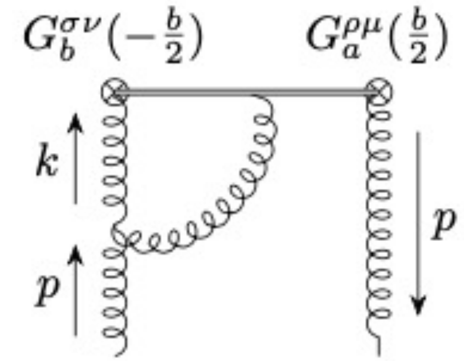
Intuition for rapidity divergences, which arise from factorization:

$$\underbrace{\int_{q_T}^{Q} \frac{\mathrm{d}k}{k}}_{\text{full}} = \lim_{\tau\to 0}\Big[\underbrace{\int_0^Q \frac{\mathrm{d}k}{k} R_c(k,\tau)}_{\text{collinear}} + \underbrace{\int_{q_T}^{\infty} \frac{\mathrm{d}k}{k} R_s(k,\tau)}_{\text{soft}}\Big] = \ln\frac{Q}{q_T}$$

Rapidity divergences and matching

Can see even at one loop. Contains terms like:

$$I = \iota^{\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{p^+ - k^+}{k^2(p-k)^2}\left(e^{i\vec{k}_T\cdot\vec{b}_T}-1\right)\left[\frac{1}{n_B\cdot k + i\delta}+\frac{1}{n_B\cdot k - i\delta}\right]$$



Collins: directly carry out the integration

LR: integrate over k^0 & k^z , get a log, then expand in $p'_z \gg k_T$ before integrating over k_T :

$$I = \frac{i}{4}\iota^{\epsilon}\int \frac{d^{d-2}k_T}{(2\pi)^{d-1}}\left(e^{i\vec{k}_T\cdot\vec{b}_T}-1\right)\left[\left(\frac{2}{k_T^2}+\frac{1}{p_z'^2}\right)\frac{\ln\left(\frac{k_T^2+2p'_z\sqrt{k_T^2+p_z'^2}+2p_z'^2}{k_T^2}\right)}{\sqrt{1+k_T^2/p_z'^2}}-\frac{4}{k_T^2}\right]$$

Yield different values:

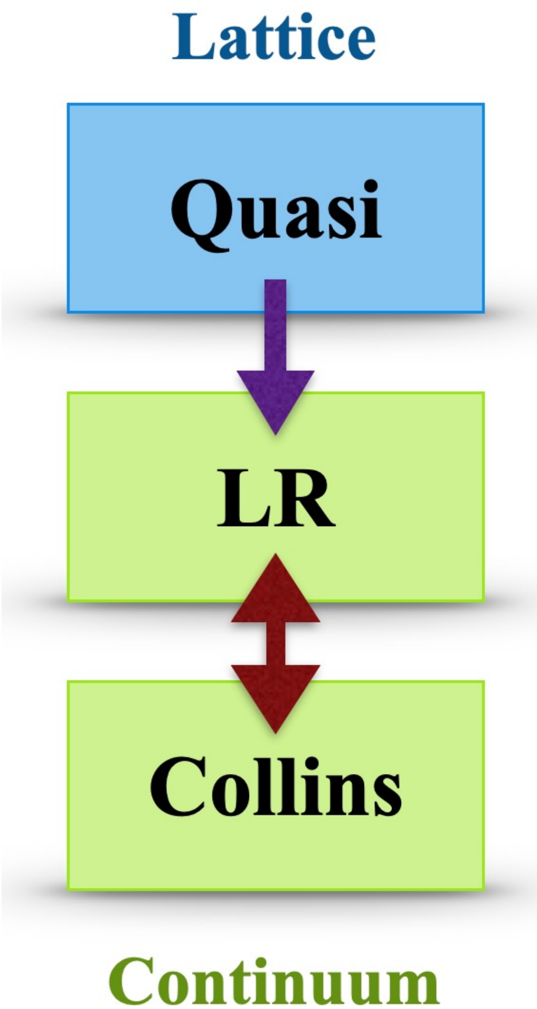
$$f_{LR} = C_i\left(x\tilde{P}^z,\mu\right)\, f_{Collins}$$

Collins $= \frac{i}{(4\pi^2)}\left[\frac{1}{\epsilon}-\frac{1}{2}\ln^2\frac{(4p'^zb_T)^2}{b_0^2}+\ln\frac{(4p'^zb_T)^2}{b_0}+\ln(\frac{b_T^2\mu^2}{b_0^2})-2\right]$

LR $= \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(2+\ln\frac{\mu^2}{4p_z'^2}\right)-\frac{1}{2}\ln^2\left(\frac{b_T^2\mu^2}{b_0^2}\right)+\ln\left(\frac{b_T^2\mu^2}{b_0^2}\right)\left(2+\ln\frac{\mu^2}{4p_z'^2}\right)-\frac{\pi^2}{12}\right]$

Schindler, Stewart, and Zhao (2022).

Factorization derivation steps



Step 1: Same at large rapidity ✅

Step 2: Pick up a matching coefficient ✅

Step 3: Combine to get full factorization

Lattice-to-physical factorization

Quasi-TMD (lattice) | **Matching** | **RGE for ζ** | **Collins TMD (continuum)**

$$\tilde{f}^{[s]}_{i/H}\left(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z\right) = C_i\left(x\tilde{P}^z, \mu\right) \exp\left[\frac{1}{2}\gamma^i_\zeta(\mu, b_T)\ln\frac{\tilde{\zeta}}{\zeta}\right] f^{[s]}_{i/H}\left(x, \vec{b}_T, \mu, \zeta\right)$$

$$\tilde{\zeta} = \left(2x\tilde{P}^z\right)^2 e^{2(y_B - y_n)}$$

Power corrections

$$\times\left\{1 + \mathcal{O}\left[\frac{1}{\left(x\tilde{P}^z b_T\right)^2}, \frac{\Lambda^2_{QCD}}{\left(x\tilde{P}^z\right)^2}\right]\right\}$$

Note that this formula connects physical continuum TMDs to the renormalized ***continuum limit*** of lattice calculations.

Ebert, **Schindler**, Stewart, and Zhao (2022).

What is the matching coefficient?

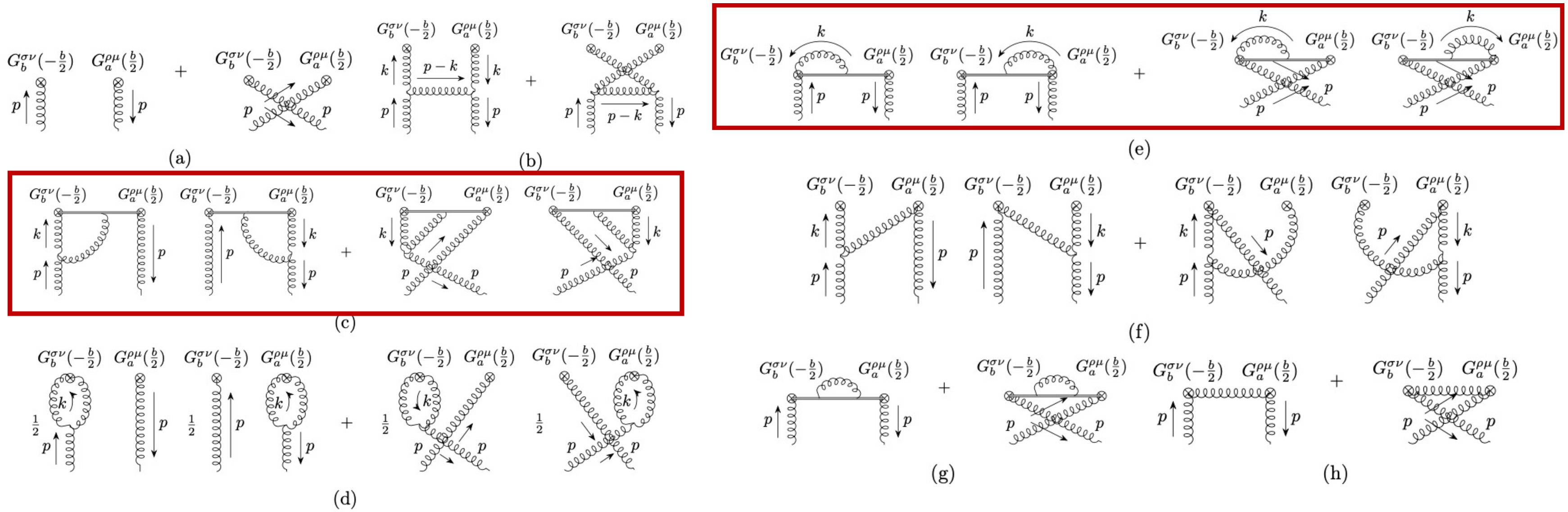
$$\tilde{f}_{i/H}^{[s]}\left(x, \vec{b}_T, \mu, \tilde{\zeta}, \mathrm{x}\tilde{P}^z\right) = \boldsymbol{C_i\left(x\tilde{P}^z, \mu\right)} \exp\left[\frac{\gamma_\zeta^i}{2} \ln\frac{\tilde{\zeta}}{\zeta}\right] f_{i/H}^{[s]}\left(x, \vec{b}_T, \mu, \zeta\right)$$

Convenient properties:

- Independent of spin
- No quark-gluon or flavor mixing
- Known at one-loop & logarithmic terms

NLO matching coefficients

Recall: only rapidity divergences contribute! For the gluon:



Casimir scaling for quarks and gluons

$$C_i\left(\mu, x\tilde{P}^z\right) = 1 + \frac{\alpha_s C_R}{4\pi}\left[-\ln^2\frac{(2xP^z)^2}{\mu^2} + \frac{2\ln(2xP^z)^2}{\mu^2} - 4 + \frac{\pi^2}{6}\right] + O(\alpha_s^2)$$

Schindler, Stewart, and Zhao, 2205.12369.

Note that Casimir scaling only holds if one chooses $F^{+\rho}$ rather than $F^{0\rho}$ or $F^{3\rho}$ (Zhang et al., 2209.05443)

N$^{\text{n}}$LL terms

RG evolution:

$$\frac{d}{d\ln(2x\tilde{P}^z)}\ln \boldsymbol{C_q(x\tilde{P}^z,\mu)} = \gamma_C^q(2x\tilde{P}^z,\mu)$$

Turn the crank, get matching coefficient:

$$C_i(x\tilde{P}^z,\mu) = C_i[\alpha_s(\mu)]\exp\left[\int_{\alpha_s(\mu)}^{\alpha_s(2x\tilde{P}^z)}\frac{d\alpha}{\beta[\alpha]}\int_{\alpha}^{\alpha_s(\mu)}\frac{d\alpha'}{\beta[\alpha']}(2\boldsymbol{\Gamma^i_{cusp}[\alpha']}+\boldsymbol{\gamma^i_C[\alpha]})\right]$$

→ N$^{\text{n}}$LL straightforward to compute from higher-order anomalous dimensions.

Example, NLL:

$$\boldsymbol{C_q(x\tilde{P}^z,\mu)}^{\boldsymbol{NLL}} = -2K_\Gamma^q(2x\tilde{P}^z,\mu) - K_\gamma^q(2x\tilde{P}^z,\mu)$$

$$K_\Gamma^q(\mu_0,\mu) = -\frac{\Gamma_0^q}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s(\mu_0)}\left(1-\frac{1}{r}-\ln r\right)+\left(\frac{\Gamma_1^q}{\Gamma_0^q}-\frac{\beta_1}{\beta_0}\right)(1-r+\ln r)+\frac{\beta_1}{2\beta_0}\ln^2 r\right\} \qquad K_\gamma^q(\mu_0,\mu) = -\frac{\gamma_{C0}^q}{2\beta_0}\ln r$$

Ebert, **Schindler**, Stewart, and Zhao (2022).

Spin independence

$$\text{Beam} = \left\langle P \left| \bar{q}_i \frac{\Gamma}{2} W_{\sqsupset}^{F}(b, \eta v, \delta) q_i \right| P \right\rangle$$

$$\Gamma \in \{\bar{\slashed{n}}, \bar{\slashed{n}}\gamma_5, i\sigma^{\alpha -}\gamma_5\}$$

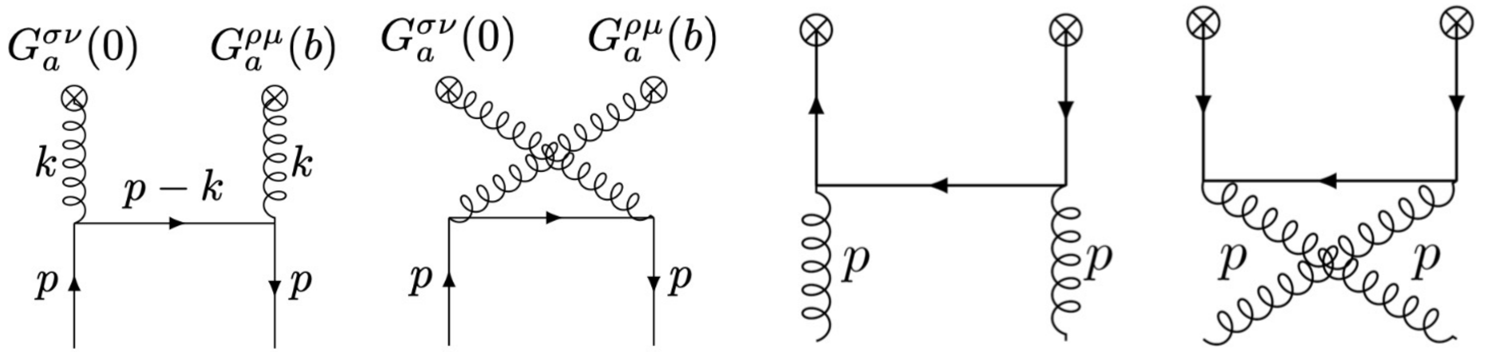
Spin structure example:

$$f_{q/h_S}^{[\slashed{n}]}(x, \vec{q}_T) = f_1(x, q_T) - \frac{\epsilon_{\rho\sigma} q_\perp^\rho S_\perp^\sigma}{M} f_{1T}^\perp(x, q_T)$$

		Quark Polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	f_1 = Unpolarized		$h_1^\perp$ = − Boer-Mulders
	L		g_1 = − Helicity	$h_{1L}^\perp$ = − Worm-gear
	T	$f_{1T}^\perp$ = − Sivers	$g_{1T}^\perp$ = − Worm-gear	h_1 = − Transversity $h_{1T}^\perp$ = − Pretzelosity

Ebert, **Schindler**, Stewart, and Zhao (2020).

No flavor or quark-gluon mixing



The diagrams above are the same for quasi, LR, and Collins:

- Can see directly from factorization derivation
- So, only two coefficients C_q & C_g
- Can do gluon TMDs!

Ebert, **Schindler**, Stewart, and Zhao (2022).

Status of the lattice

Lattice targets

$$d\sigma = H \int f \otimes f$$

Full TMDs (hadrons, flavors, spins)

$$f = Z_{UV} \frac{B}{\sqrt{S}}$$

Beam functions (& ratios)

Soft factor (indirect)

$$f = C \times \tilde{f}_{lattice}$$

CS kernel (ζ evolution)

CS kernel from beam ratios

From the factorization formula:

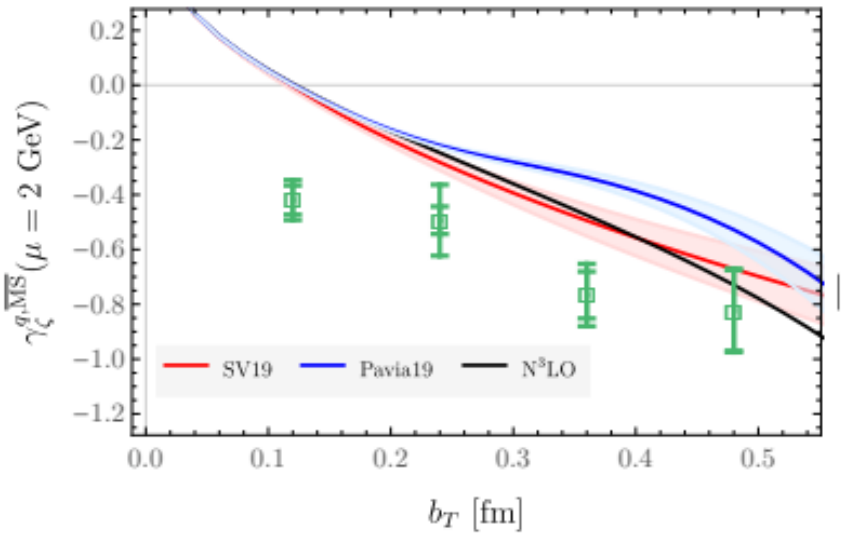
$$\gamma_\zeta^q(\mu, b_T) = \frac{1}{\ln P_1^z/P_2^z} \ln \frac{C^{TMD}(\mu, xP_2^z)\int db^z e^{ib^z xP_1^z}\tilde{Z}'_q \tilde{Z}^q_{uv}\tilde{B}_q(b^z, \vec{b}_T, a, L, P_1^z)}{C^{TMD}(\mu, xP_1^z)\int db^z e^{ib^z xP_2^z}\tilde{Z}'_q \tilde{Z}^q_{uv}\tilde{B}_q(b^z, \vec{b}_T, a, L, P_2^z)}$$

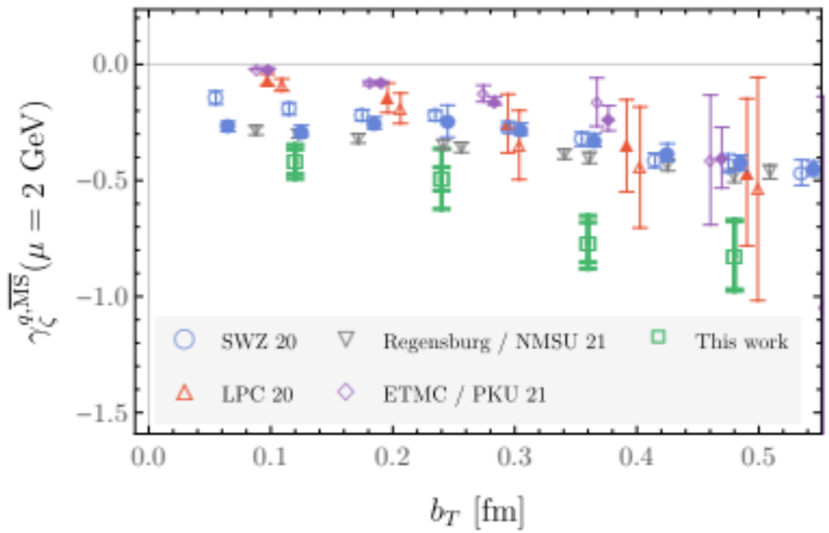
Dependent on few parameters compared to RHS!

- No soft function needed
- Can set up $\tilde{Z}^q_{uv}$ to remove power law divergences in numerator and denominator

Ebert, Stewart, and Zhao (PRD 2019).

CS kernel lattice results





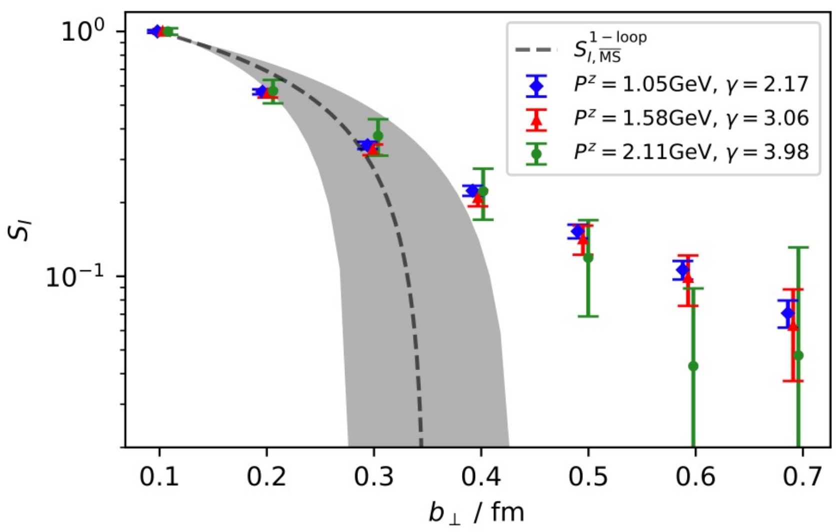
Recent first lattice results!

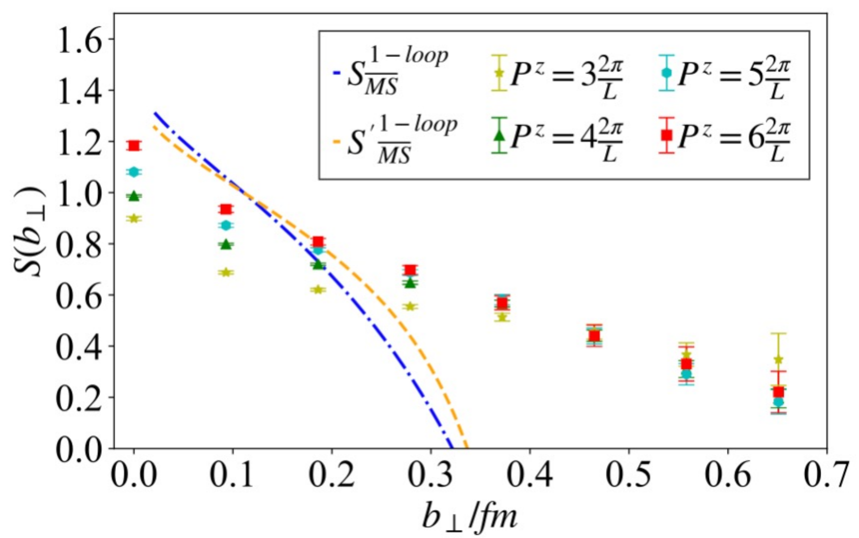
However, large systematic uncertainties.

Shanahan, Wagman, Zhao (PRD 2021).

Soft function on the lattice

- Soft function also runs into lightcone Wilson staple issues
- Can express soft as ratio of meson form factor with convolution of two meson wavefunctions





Ji, Liu, Liu (NPB 2020). LPC collaboration (PRL 2020). Li et al. (PRL 2022).

TMD ratios from beam ratios

Ratios of different TMD spins, flavors, or hadrons can be calculated directly from lattice beam functions:

$$\lim_{\tilde{\eta}\to\infty} \frac{f_{q_i/h}^{[\tilde{\Gamma}_1]}}{f_{q_j/h'}^{[\tilde{\Gamma}_2]}} = \lim_{\tilde{\eta}\to\infty} \frac{\tilde{B}_{q_i/h}^{[\tilde{\Gamma}_1]}}{\tilde{B}_{q_j/h'}^{[\tilde{\Gamma}_2]}}$$

This follows from the quasi-to-Collins factorization formulas:

$$C_i \exp\left[\frac{1}{2}\gamma_\zeta^i \ln\frac{\tilde{\zeta}}{\zeta}\right] f_{q_i/H}^{[\Gamma]} = \tilde{f}_{q_i/H}^{[\Gamma]} = \lim Z_{UV} \frac{\tilde{B}_{q_i/H}^{[\Gamma]}}{\sqrt{S^R}}$$

Lattice-to-continuum TMD factorization

Factorization of a lattice TMD into matrix elements

MHENS on the lattice

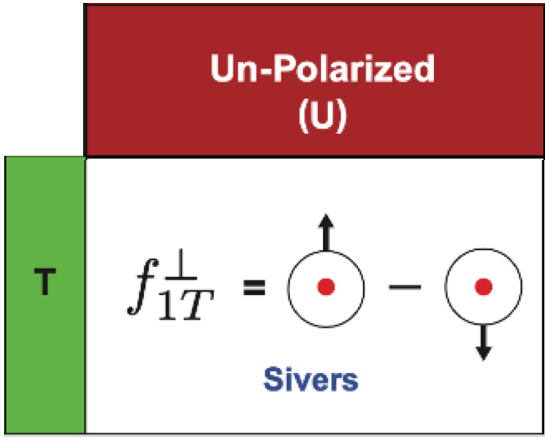
First lattice studies!

- Suggested ratio method
- Focus on *x*-integrated TMDs, so renormalization is less of a problem

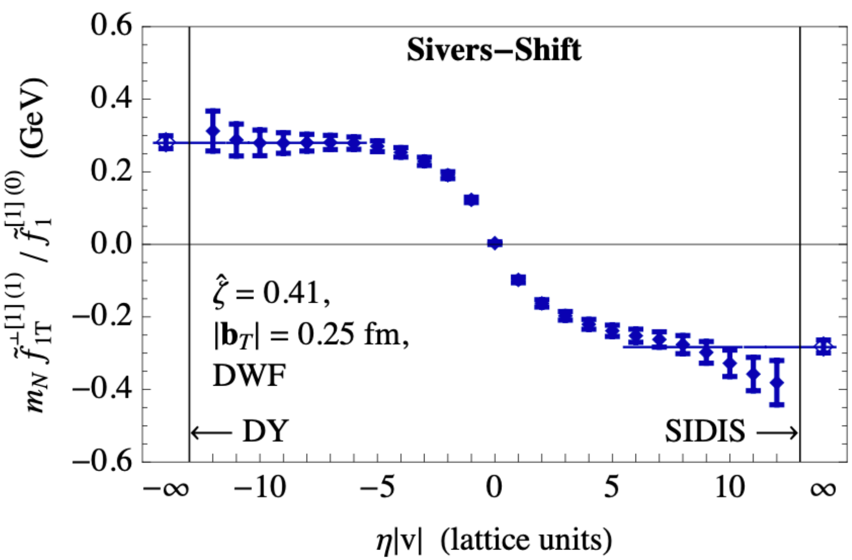
Caveat:

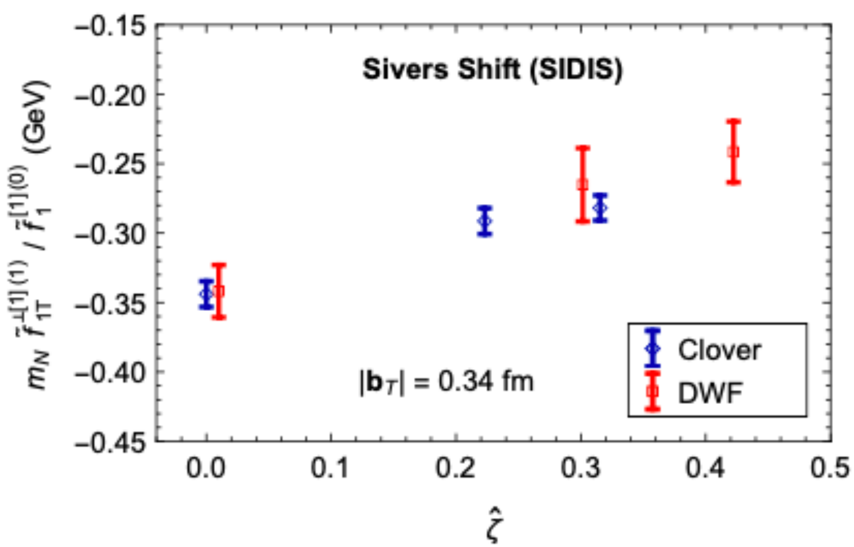
- So far, no matching corrections
- Procedure to carry out simulations with matching, x-dependence, soft functions not yet known

MHENS lattice results: Sivers sign change



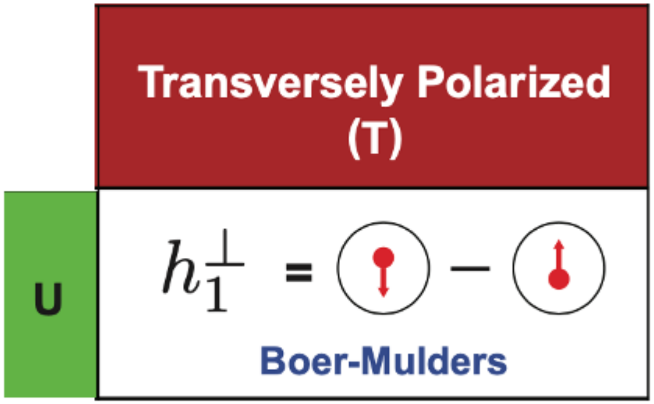
- Sivers has different signs in DY & SIDIS
- Can verify on the lattice using ratios at various b_T values





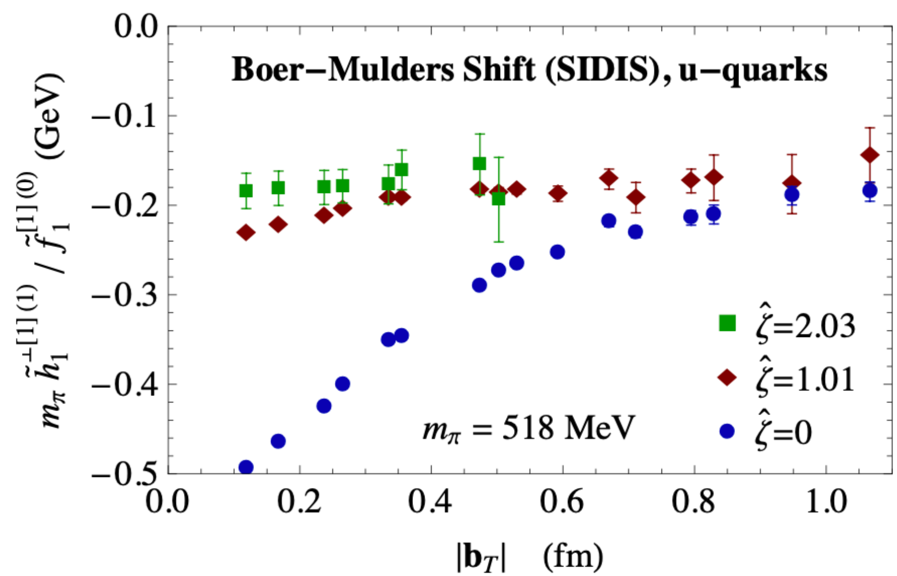
Yoon, Engelhardt, Gupta, et al. (PRD 2017).

MHENS lattice results: Boer-Mulders shift



- Pion u-quark Boer-Mulders shift in SIDIS

$$m_N \frac{\tilde{h}_1^\perp}{\tilde{f}_1}$$



MHENS (PRD 2016).

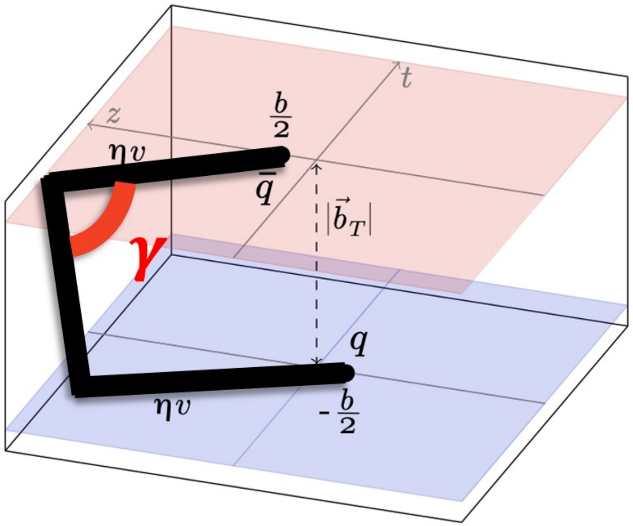
Caveat: nontrivial MHENS-to-Collins connection

For the case $P \cdot b = 0$ (focus of all studies so far) MHENS and quasi have an equivalent renormalization, soft function, etc.

$$\int \mathrm{d}x \; \tilde{f}^{[\Gamma]}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = \; f^{[\Gamma]\mathrm{MHENS}}_{q_i/h}(b^z = 0, \vec{b}_T, \mu, \tilde{P}^z, y_n - y_B, \tilde{\eta})$$

For the case $P \cdot b \neq 0$:

- Non-trivial cusp angles γ, even as $\eta \to \infty$
- b^z-dependent Wilson length
- Implies renormalization, soft are b^z-dependent and won't cancel out in ratios at finite η



Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Status of the lattice

CS kernel	⌛
Spin-dependent TMD ratios	⌛
3D structure ratios	⌛
Flavor ratios	⌛
Normalized TMD	❌
Proton-pion TMD ratios	❌
Gluon TMDs	❌
…	…

Conclusion

Implications of factorization

Quasi-to-Collins matching coefficient: quite convenient…

- No spin dependence
- No quark-gluon or flavor mixing (simpler to get gluon TMDs!)
- NLO & N^nLL results: generalized Casimir scaling
- Same as LR-to-Collins coefficients, so can compute as the rapidity-divergent diagrams in different orders of limits

Implies validity of taking quasi-TMD ratios…

- P^z ratios for CS kernel
- Beam hadron, flavor, spin ratios for full TMD ratios

Our contributions

1. New unified TMD notation
2. New scheme (LR)
3. Lattice-to-physical TMD factorization: convenient!

Quasi-TMDs have a straightforward, rigorous connection to physical TMDs